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A Computational Algorithm to Obtain Positive Solutions for Classes of Competitive Systems

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Abstract. Using a numerical method based on sub-super solution, we will obtain positive solution to the coupled-system of boundary value problems of the form

$$\begin{aligned} -\Delta u(x) &= \lambda f(x,u,v) && x\in\Omega \\ -\Delta v(x) &= \lambda g(x,u,v) && x\in\Omega \\ u(x) &= v(x) = 0 && x\in\partial\Omega \end{aligned}$$

where f,g are C^1 functions with at least one of $f(x_{0,},0,0)$ or $g(x_{0,},0,0)$ being negative for some $x_0\in\Omega$ (semipositone).

Keywords: positive solutions; sub and super-solutions
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1. Introduction

Consider positive solutions to the coupled-system of boundary value problems

$$\begin{aligned} -\Delta u(x) &= \lambda f(x,u,v) && x\in\Omega \\ -\Delta v(x) &= \lambda g(x,u,v) && x\in\Omega \\ u(x) &= v(x) = 0 && x\in\partial\Omega \end{aligned} \tag{1}$$

Where $\lambda > 0$ is a parameter, Δ is the Laplacian operator, Ω is a bounded region in R^N, $N\ge 1$ with a smooth boundary $\partial\Omega$, and f,g are C^1 functions with at least one of $f(x_{0,},0,0)$ or $g(x_{0,},0,0)$ being negative for some $x_0\in\Omega$ (semipositone).

In this paper, we want to investigate numerically positive solution of (1) by using the method of sub-super solutions. A super solution to (1) is defined as an ordered pair of smooth functions $(\overline{u},\overline{v})$ on Ω satisfying

$$\begin{aligned} -\Delta \overline{u}(x) &\ge \lambda f(x,\overline{u},\overline{v}) && x\in\Omega \\ -\Delta \overline{v}(x) &\ge \lambda g(x,\overline{u},\overline{v}) && x\in\Omega \\ \overline{u}(x) \ge 0;\overline{v}(x) &\ge 0; && x\in\partial\Omega. \end{aligned} \tag{2}$$

Sub solutions are similarly defined with inequalities reversed. Let $D=[\underline{\rho}_1,\overline{\rho}_1]\times[\underline{\rho}_2,\overline{\rho}_2]$, where

$\underline{\rho}_1=\inf\{\underline{u}(x):x\in\overline{\Omega}\}, \overline{\rho}_1=\sup\{\overline{u}(x):x\in\overline{\Omega}\}, \underline{\rho}_2=\inf\{\underline{v}(x):x\in\overline{\Omega}\}, \overline{\rho}_2=\sup\{\overline{v}(x):x\in\overline{\Omega}\}$.

Theorem 1. Let $(\overline{u},\overline{v})$, $(\underline{u},\underline{v})$ be ordered pairs of smooth functions such that $(\overline{u},\underline{v})$ satisfies

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$$
\begin{aligned}
&-\Delta\overline{u}(x) \geq \lambda f(x,\overline{u},\underline{v}) && x \in \Omega \\
&-\Delta\underline{v}(x) \leq \lambda g(x,\overline{u},\underline{v}) && x \in \Omega \\
&\overline{u}(x) \geq 0; \underline{v}(x) \leq 0; && x \in \partial\Omega.
\end{aligned}
$$

And $(\underline{u},\overline{v})$ satisfies the corresponding reserved inequalities. Suppose that

$$\frac{\partial f}{\partial v},\frac{\partial g}{\partial u} \leq 0 \quad \text{on} \quad \overline{\Omega}\times D \quad \text{(cooperative system).}$$

If $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ on $\overline{\Omega}$, then there is a solution (u,v) of (1) such that $\underline{u} \leq u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$ on Ω.

In [1] for the first time in the literature, the authors consider a class of semipositone systems. In particular they extend many of the results discussed for the positive solutions of single equation in [2] to semipositone systems. It was shown positive solutions to (1) for either λ near the first eigenvalue λ_1 of the operator $-\Delta$ subject to Dirichlet boundary conditions, or for λ large exists. We consider following assumptions:

f, g are C^1 functions satisfying :

$$\text{either} \quad f(x_{0,},0,0) < 0 \ \text{ or } \ g(x_{0,},0,0) < 0 \ \text{ for some } \ x_0 \in \Omega \tag{3}$$

$$\lim_{u\to\infty}\frac{f(x,u,v)}{u} = 0 \quad \text{uniformly in } x,v \tag{4}$$

and

$$\lim_{v\to\infty}\frac{g(x,u,v)}{v} = 0 \quad \text{uniformly in } x,u \tag{5}.$$

To introduce additional hypotheses to prove existence results near λ_1,first we recall the anti-maximum principal by Clement Pletier(see [4]), namely, if z_λ is the unique solution of

$$
\begin{aligned}
&-\Delta z - \lambda z = -1 && x \in \Omega \\
&z = 0 && x \in \partial\Omega
\end{aligned}
\tag{6}
$$

for $(\lambda_1, \lambda_1 + \delta)$, where λ_1 is the smallest eigenvalue of the problem

$$
\begin{aligned}
&-\Delta\phi(x) = \lambda\phi(x) && x \in \Omega \\
&\phi(x) = 0 && x \in \partial\Omega.
\end{aligned}
\tag{7}
$$

Let $I = [\alpha,\gamma]$ where $\alpha > \lambda_1$ and $\gamma < \lambda_1 + \delta$, and let

$$\sigma := \max_{\lambda\in I} \| z_\lambda \|$$

Where $\|.\|$ denotes the supremum norm. Now assuming that there exists a $m_1 > 0$ such that

$$f(x,u,v) \geq u - m_1 \qquad \forall x \in \overline{\Omega}, u \in [0, m_1\gamma\sigma], \quad v \geq 0 \tag{8}$$

and exists a $m_2 > 0$ such that

$$g(x,u,v) \geq v - m_2 \qquad \forall x \in \overline{\Omega}, v \in [0, m_2\gamma\sigma], \quad u \geq 0 \ . \tag{9}$$

Finally to prove existence results for λ large, in addition to (3)-(5), we assume $\exists f_1(u) \leq f(x,u,v)$ $\forall x \in \overline{\Omega}, u \geq 0, v \geq 0$ such that $f_1(r_1) = 0,\ f_1'(r_1) < 0,$

$$\int_0^{r_1} f_1(s)ds > 0 \quad \text{for some } r_1 > 0 \tag{10}$$

And $\exists g_2(v) \leq g(x,u,v)$ $\forall x \in \overline{\Omega}, u \geq 0, v \geq 0$ such that $g_2(r_2) = 0, g_2'(r_2) < 0,$

$$\int_0^{r_2} g_2(s)ds > 0 \quad \text{for some } r_2 > 0 \tag{11}$$

2. Existence results

Theorem 2. Let $\lambda_1 \in I$ and assume (3)-(5), and (8)-(9) hold, Then (1) has a positive solution.

It was shown in [1] $(\underline{u}, \underline{v})$ is a subsolution of (1) where $\underline{u}(x) = \gamma m_1 z_\lambda$ and $\underline{v}(x) = \gamma m_2 z_\lambda$.

Now let $w(x)$ to be the unique positive solution of

$$\begin{aligned} -\Delta w(x) &= 1 \qquad && x \in \Omega \\ w(x) &\le 0 && x \in \partial\Omega. \end{aligned} \tag{12}$$

$(\overline{u}, \overline{v})$ is a supersolution that $\overline{u} = Jw(x)$ and $\overline{v} = \tilde{J}w(x)$ where $J, \tilde{J} > 0$, are sufficiently large, such that

$$\frac{1}{\lambda \| w \|} \ge \frac{f(x, J \| w \|, v)}{J \| w \|}, \frac{g(x, u, \tilde{J} \| w \|)}{\tilde{J} \| w \|} \tag{13}$$

and

$$\overline{u}(x) \ge \underline{u}(x) \text{ on } \Omega \text{ and } \overline{v}(x) \ge \underline{v}(x) \text{ on } \Omega \tag{13$'$}$$

Theorem 3. Assume (3)-(5) and (10)-(11) hold. Then there exists a $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, (1) has a positive solution.

Here we give a simple example that satisfies the hypotheses of theorem 2 and 3. Consider

$$h(x,u,v) = m\sqrt{u+1} - \frac{3m}{2} + e^{-v} \qquad \forall u \ge 0, v \ge 0 \tag{14}$$

where m > 0 is a constant. Let

$$f(x,u,v) = h(x,u,v)$$
$$g(x,u,v) = h(x,u,v)$$

Here $f(x,0,0) = 1 - \frac{m}{2} < 0$ for $m > 2$, f is increasing in u, v, and $\lim_{u\to\infty} \frac{f(x,u,v)}{u} = 0$ uniformly in v.

Also $g(x,0,0) = 1 - \frac{m}{2} < 0$ for $m > 2$, g is increasing in u, v, and $\lim_{v\to\infty} \frac{g(x,u,v)}{v} = 0$ uniformly in u.

Now, to show that (8) and (9) are satisfied, it suffices to show that $h_1(u) = m\sqrt{u+1} - \frac{3m}{2}$ satisfies (8) since $h(x,u,v) \ge h_1(u) \forall u \ge 0, v \ge 0$. Let $p > 0$ be such that $h_1(p) = p - m$. That is,

$$m\sqrt{p+1} - \frac{3m}{2} = p - m,$$

$$m^2(p+1) = \left\{ p + \frac{m}{2} \right\}^2,$$

$$p^2 + (m - m^2)p - \frac{3m^2}{4} = 0,$$

and

$$p = \frac{(m^2 - m) + \sqrt{m^4 - 2m^3 + 4m^2}}{2}$$
$$= \frac{(m^2 - m) + m\sqrt{(m-1)^2 + 3}}{2}.$$

Hence in order that (8) be satisfied, we must have

$$\frac{m^2 - m + m\sqrt{(m-1)^2 + 3}}{2} \geq m(\sigma\alpha),$$

that is,

$$(m-1) + \sqrt{(m-1)^2 + 3} \geq 2(\sigma\alpha) \qquad (15)$$

Since σ and α are quantities that depend only on Ω, clearly for a given σ and α, there exists and m_0 sufficiently large such that if $m > m_0$, then (15) is satisfied and equivalently (8) will be satisfied. Thus, (9) is also satisfied for $m > m_0$.

Note that this example satisfies the hypotheses of theorem 3 also since $h(x,u,v) \geq h_1(u) \forall u \geq 0, v \geq 0$ and one can construct a function $f_1(u) \leq h_1(u)$ satisfying (10).

3. Numerical Results

We see in section 2 that there must always exists a solution for problems such as (1) between a sub-solution $(\underline{u},\underline{v})$ and a super-solution $(\overline{u},\overline{v})$ when $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \leq 0$

Consider the coupled-system boundary value problems

$$\begin{aligned} -\Delta u(x) &= \lambda f(x,u,v) && x \in \Omega \\ -\Delta v(x) &= \lambda g(x,u,v) && x \in \Omega \\ u(x) &= v(x) = 0 && x \in \partial\Omega \end{aligned} \qquad (16)$$

Since f, g are C^1 functions, there exists positive constants k_1, k_2 such that $\frac{\partial f}{\partial u} \geq -k_1$, and $\frac{\partial g}{\partial v} \geq -k_2$ on $\overline{\Omega} \times D$. Thus we can study the equivalent system

$$\begin{aligned} -\Delta u(x) + \lambda k_1 u(x) &= \lambda f(x,u,v) + \lambda k_1 u(x) = \lambda \hat{f}(x,u,v) && x \in \Omega \\ -\Delta v(x) + \lambda k_2 v(x) &= \lambda g(x,u,v) + \lambda k_2 v(x) = \lambda \hat{g}(x,u,v) && x \in \Omega \\ u(x) &= v(x) = 0 && x \in \partial\Omega \end{aligned} \qquad (17)$$

The mapping $T : (u_1, v_1) \to (u_2, v_2),\ (u_2, v_2) = T(u_1, v_1)$

$$(u_1, v_1) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] \qquad \forall x \in \overline{\Omega}$$

Where (u_2, v_2) is the unique solution of the coupled-system

$$\begin{aligned} -\Delta u_2(x) + \lambda k_1 u_2(x) &= \lambda f(x,u_1,v_1) + \lambda k_1 u_1(x) && x \in \Omega \\ -\Delta v_2(x) + \lambda k_2 v_2(x) &= \lambda g(x,u_1,v_1) + \lambda k_2 v_1(x) && x \in \Omega \\ u_2(x) &= v_2(x) = 0 && x \in \partial\Omega \end{aligned} \qquad (18)$$

satisfied the hypotheses of Schauder fixed point theorem, and then we can conclude that

$$\exists (u,v) \in D \qquad T(u,v) = (u,v)$$

so (u,v) is a solution of (1) (see [3]).

By letting $\hat{f}(x,u,v) = \lambda f(x,u,v) + \lambda k_1 u(x)$ and $\hat{g}(x,u,v) = \lambda g(x,u,v) + \lambda k_2 v(x)$, we use the following iteration to obtain solution:

$$\begin{aligned} u_0(x) &= \underline{u}, v_0(x) = \overline{v} && n = 0,1,2,\ldots \\ (\Delta - \lambda k_1) u_{n+1} &= -\hat{f}(x, u_n, v_n) && x \in \Omega \\ (\Delta - \lambda k_2) v_{n+1} &= -\hat{g}(x, u_{n+1}, v_n) && x \in \Omega \\ u_{n+1} &= 0 = v_{n+1} && x \in \partial\Omega. \end{aligned} \qquad (19)$$

We can also use $u_0(x)=\overline{u}, v_0(x)=\underline{v}$ as initial guesses. we use following algorithm

sub- and super-solution algorithm

1. Find $u_0(x)=\underline{u}, v_0(x)=\overline{v}$. Choose numbers $k_1, k_2>0$;
2. Solve the boundary value system (19);
3. If $\| u_{n+1}-u_n \|<\varepsilon$ and $\| v_{n+1}-v_n \|<\varepsilon$, output and stop. Else go to step 2.

Now we want to apply the algorithm for:

$$\begin{aligned} &-\Delta u(x)=\lambda(m\sqrt{u+1}-\frac{3m}{2}+e^{-v}) && x\in\Omega \\ &-\Delta v(x)=\lambda(m\sqrt{v+1}-\frac{3m}{2}+e^{-u}) && x\in\Omega \\ &u(x)=v(x)=0 && x\in\partial\Omega \end{aligned} \tag{20}$$

For doing step 1, we solve the problem

$$\begin{aligned} &-\Delta z-\lambda z=-1 && x\in\Omega \\ &z=0 && x\in\partial\Omega \end{aligned} \tag{21}$$

to obtain $\underline{u}$. We know from section 2 that problem (21) has a positive solution for $(\lambda_1,\lambda_1+\delta)$. The obtained results show there is an array of positive solution for $\lambda\in(17,35)$ so λ_1 is around 17.

For brevity we express just some of those numerical results:

Approximation of z_λ for $\lambda=15$

x/y	0.2	0.4	0.6	0.8
0.2	-0.268	0.423	-0.431	-0.283
0.4	-0.447	-0.701	-0.718	-0.493
0.6	-0.513	-0.753	-0.778	-0.636
0.8	-0.505	-0.528	-0.497	-0.345

Approximation of z_λ for $\lambda=17$

x/y	0.2	0.4	0.6	0.8
0.2	1.895	3.067	3.130	2.022
0.4	3.266	5.199	5.341	3.625
0.6	3.789	5.626	5.818	4.172
0.8	3.727	3.912	3.676	2.514

Approximation of z_λ for $\lambda=30$

x/y	0.2	0.4	0.6	0.8
0.2	0.002	0.017	0.021	0.008
0.4	0.028	0.062	0.068	0.041
0.6	0.050	0.087	0.093	0.070
0.8	0.054	0.060	0.056	0.033

Approximation of z_λ for $\lambda=36$

x/y	0.2	0.4	0.6	0.8
0.2	-0.001	0.012	0.016	0.005
0.4	0.024	0.056	0.063	0.038

0.6	0.048	0.084	0.091	0.068
0.8	0.053	0.060	0.056	0.033

Let $\underline{u} = \gamma m_{1} z_{\lambda}(x)$ where γ and m obtained from section 1, 2 and to obtain $\overline{v}$ for $\lambda \in I$ ($I = [\alpha, \gamma]$ where $\alpha > \lambda_1$ and $\gamma < \lambda_1 + \delta$) we solve

$$\begin{aligned} -\Delta v(x) &= 1 \qquad x \in \Omega \\ v(x) &= 0 \qquad x \in \partial\Omega \end{aligned} \tag{22}$$

by finite difference (see [5,6]). We choose J such that (13) , $(13)'$ are satisfied.

We execute algorithm for $\lambda \in [17.1, 34.9]$. It is easy to see that $u = v$ for problem (20).

For brevity we express just some of those numerical results:

Approximation of u for $\lambda = 17.1$

x/y	0.2	0.4	0.6	0.8
0.2	1.009×10^4	1.510×10^4	1.510×10^4	1.009×10^4
0.4	1.510×10^4	2.777×10^4	2.777×10^4	1.510×10^4
0.6	1.510×10^4	2.777×10^4	2.777×10^4	1.510×10^4
0.8	1.009×10^4	1.510×10^4	1.510×10^4	1.009×10^4

Approximation of u for $\lambda = 25$

x/y	0.2	0.4	0.6	0.8
0.2	2.173×10^4	3.253×10^4	3.253×10^4	2.173×10^4
0.4	3.253×10^4	4.902×10^4	4.902×10^4	3.253×10^4
0.6	3.253×10^4	4.902×10^4	4.902×10^4	3.253×10^4
0.8	2.173×10^4	3.253×10^4	3.253×10^4	2.173×10^4

Approximation of u for $\lambda = 30$

x/y	0.2	0.4	0.6	0.8
0.2	3.139×10^4	4.697×10^4	4.697×10^4	3.139×10^4
0.4	4.697×10^4	7.078×10^4	7.078×10^4	4.697×10^4
0.6	4.697×10^4	7.078×10^4	7.078×10^4	4.697×10^4
0.8	3.139×10^4	4.697×10^4	4.697×10^4	3.139×10^4

Approximation of u for $\lambda = 34.9$

x/y	0.2	0.4	0.6	0.8
0.2	4.256×10^4	6.368×10^4	6.368×10^4	4.256×10^4
0.4	6.368×10^4	9.596×10^4	9.596×10^4	6.368×10^4
0.6	6.368×10^4	9.596×10^4	9.596×10^4	6.368×10^4
0.8	4.256×10^4	6.368×10^4	6.368×10^4	4.256×10^4

Our numerical results (in following tables) show that there exist $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, (20) has a positive solution. In this case $\lambda^* = 166.696$ with decimal accuracy.

Approximation of u for $\lambda = 170$

x/y	0.2	0.4	0.6	0.8
0.2	1.019×10^{6}	1.524×10^{6}	1.524×10^{6}	1.019×10^{6}
0.4	1.524×10^{6}	2.297×10^{6}	2.297×10^{6}	1.524×10^{6}
0.6	1.524×10^{6}	2.297×10^{6}	2.297×10^{6}	1.524×10^{6}
0.8	1.019×10^{6}	1.524×10^{6}	1.524×10^{6}	1.019×10^{6}

Approximation of u for $\lambda = 500$

x/y	0.2	0.4	0.6	0.8
0.2	0.883×10^{7}	1.321×10^{7}	1.321×10^{7}	0.883×10^{7}
0.4	1.321×10^{7}	1.990×10^{7}	1.990×10^{7}	1.321×10^{7}
0.6	1.321×10^{7}	1.990×10^{7}	1.990×10^{7}	1.321×10^{7}
0.8	0.883×10^{7}	1.321×10^{7}	1.321×10^{7}	0.883×10^{7}

Approximation of u for $\lambda = 1000$

x/y	0.2	0.4	0.6	0.8
0.2	3.534×10^{7}	5.286×10^{7}	5.286×10^{7}	3.534×10^{7}
0.4	5.286×10^{7}	7.963×10^{7}	7.963×10^{7}	5.286×10^{7}
0.6	5.286×10^{7}	7.963×10^{7}	7.963×10^{7}	5.286×10^{7}
0.8	3.534×10^{7}	5.286×10^{7}	5.286×10^{7}	3.534×10^{7}

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