

Exact Solutions for Nonlinear PDEs with the Variable Coefficients in Mathematical Physics

Khaled A. Gepreel⁺

Mathematics Department, Faculty of Science, Taif University, El-Taif, El-Hawiyah, P.O.Box 888, Kingdom of Saudi Arabia.

Mathematics Department, Faculty of Science, Zagazig University, Egypt.

(Received October 19, 2010, accepted December 20, 2010)

Abstract. In this article, we construct the exact solutions for nonlinear partial differential equations with the variable coefficients in the mathematical physics via the generalized time-dependent variable coefficients KdV-mKdV equation and the coupled modified KdV equations with non-uniformity terms by using a generalized (G'/G) -expansion method with the variable coefficients, where G satisfies the Jacobi elliptic equation. Many of the exact solutions in terms of Jacobi elliptic functions are obtained. The proposed method is reliable and effective and gives more new exact solutions.

Keywords: A generalized (G'/G) -expansion method with the variable coefficients, The generalized time-dependent variable coefficients KdV- mKdV equation, The coupled modified KdV equations with non-uniformity terms, The Jacobi elliptic functions.

AMS Subject Classifications: 35K99; 35P05; 35P99.

1. Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors (see for example [1-36]) who are interested in nonlinear physical phenomena. Many powerful different methods have been presented by those authors. For integrable nonlinear differential equations, the inverse scattering transform method [3], the Hirota method [8], the truncated Painleve expansion method [23,30], the Backlund transform method [14,15] and the exp- function method [5,32] are used to find the exact solutions. Among non-integrable nonlinear differential equations there is a wide class of equations that referred to as partially integrable, because these equations become integrable for some values of their parameters. There are many different methods used to find the exact solutions of these equations. The most famous algorithms are the tanh- function method [1,7,33], the Jacobi elliptic function expansion method [6,11,13,26,27], F- expansion method [2,18,31] and the generalized Riccati equation [17]. There are other methods which can be found in [10,12,17-25].

Wang et.al.[22] have introduced a simple method which is called the (G'/G) -expansion method to look for traveling wave solutions of nonlinear evolution equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are arbitrary constants. For further references see the articles [4,28,34,35]. Recently, Zayed [29] introduced an alternative approach, which is called a generalized (G'/G) -expansion method, where $G = G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$, $\xi = x - Vt$ and e_0, e_1, e_2, V are arbitrary constants while $' = \frac{d}{d\xi}$. The main objective of this article is using the generalized (G'/G) -expansion method with

the variable coefficients to construct the exact solutions for nonlinear evolution equations in the mathematical physics via the generalized time-dependent variable coefficients KdV- mKdV equation and the coupled modified KdV equations with non-uniformity terms, where G satisfies the Jacobi elliptic

⁺ Corresponding author. E-mail address: kagepreel@yahoo.com

equation. Many exact solutions in terms of Jacobi elliptic functions are obtained.

2. Description of a generalized (G'/G) - expansion method with the variable coefficients

Suppose we have the following nonlinear partial differential equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0, \quad (2.1)$$

where $u = u(x, t)$ is an unknown function, F is a polynomial in $u = u(x, t)$ and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of a generalized (G'/G) - expansion method

Step 1. Suppose the solution of Eq.(2.1) can be expressed by a polynomial in (G'/G) as follows

$$u(\xi) = \sum_{i=0}^n \alpha_i(t) \left(\frac{G'(\xi)}{G(\xi)} \right)^i, \quad (2.2)$$

where $\xi = p(t)x + q(t)$ and $\alpha_i(t)$ ($i = 0, 1, 2, \dots, n$), $p(t)$, $q(t)$ are arbitrary functions of t to be determined later while $G = G(\xi)$ satisfies the following Jacobi elliptic equation:

$$G''(\xi) = 2e_2 G^3(\xi) + e_1 G(\xi) \quad (2.3)$$

or

$$[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$$

and e_2, e_1, e_0 are arbitrary constants.

Step 2. The positive integer " n " can be determined by considering the homogeneous balance between the highest order partial derivative and the nonlinear terms appearing in Eq. (2.1). Therefore, we can get the value of n in (2.2).

Step 3. Substituting (2.2) into (2.1) with the conditions (2.4), we obtain polynomial in $x^i (G'(\xi))^j G^k(\xi)$, ($i, j = 0, 1, k = 0, \pm 1, \pm 2, \dots$). Equating each coefficient of the resulted polynomial to zero, yields a set of ordinary differential equations $\alpha_i(t)$, ($i = 0, 1, \dots, n$), $p(t)$ and $q(t)$.

Step 4. Solving the obtained system of the differential equations with the aid of Maple or mathematica to calculate $\alpha_i(t)$, ($i = 0, 1, \dots, n$), $p(t)$ and $q(t)$.

Step 5. Since the general solutions of Eq. (2.3) have been well known for us (see Appendix A), then substituting $\alpha_i(t)$, ($i = 0, 1, \dots, n$), $p(t)$, $q(t)$ and the general solution of Eq. (2.3) into (2.2) we have many new exact solutions of the nonlinear partial differential equation (2.1).

3. Some applications of the generalized (G'/G) - expansion method

In this section, we apply the generalized (G'/G) - expansion method with the variable coefficients to construct the exact solutions for the generalized time- dependent variable coefficients KdV- mKdV equation and the coupled modified KdV equations with non-uniformity terms which are very important nonlinear evolution equations in mathematical physics.

Example 1. The generalized time- dependent variable coefficients KdV- mKdV equation

We start with the generalized time- dependent variable coefficients KdV- mKdV equation[16] in the following form:

$$u_t - 6f_0(t)uu_x - 6f_1(t)u^2u_x + f_2(t)u_{xxx} - f_3(t)u_x + f_4(t)(Au + xu_x) = 0, \quad (3.1)$$

where $f_0(t), f_1(t), f_2(t), f_3(t), f_4(t)$ are arbitrary functions of t and A is a constant. This equation describes the propagation of weakly nonlinear waves in a KdV- typed medium that is characterized by a varying dispersion and nonlinear coefficients. Suppose that the solution of Eq. (3.1) can be expressed by a polynomial in (G'/G) as Eq.(2.2). Considering the homogeneous balance between the highest order partial derivative u_{xxx} and the nonlinear term uu_x in (3.1), we deduce that $n = 2$. Thus, the exact solution