

# Exact and Numerical Solution of Lienard's Equation by the Variational Homotopy Perturbation Method

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**Abstract.** In this paper, exact and numerical solutions are obtained for the Lienard's equation by variational homotopy perturbation method (VHPM). Comparisons are made among the variational iteration method (VIM), the exact solutions and the proposed method. The results reveal that the proposed method is very effective and simple and can be applied for other nonlinear problems in mathematical.

**Keywords:** Variational Homotopy Perturbation Method, Lagrange multiplier, Lienard's equation.

## 1. Introduction

In this paper, we consider Lienard equation:

$$x'' + f(x)x' + g(x) = h(t), \quad (1)$$

which is not only regarded as a generalization of the damped pendulum equation or a damped spring-mass system (where  $f(x)x'$  is the damping force,  $g(x)$  is the restoring force and  $h(t)$  is the external force), but also used as nonlinear models in many physically significant fields when taking different choices for  $f(x)$ ,  $g(x)$  and  $h(t)$ . For example, the choices  $f(x) = \varepsilon(x^2 - 1)$ ,  $g(x) = x$  and  $h(t) = 0$  lead equation of (1) to the Van der Pol equation served as a nonlinear model of electronic oscillation. Therefore, studying equation of (1) is of physical significance. In the general case, it is commonly believed that it is very difficult to find its exact solution by usual ways [3]. The following special case of equation (1) was studied in [1, 2, 3]:

$$x'' + lx + mx^3 + nx^5 = 0, \quad (2)$$

where  $l$ ,  $m$  and  $n$  are real coefficients. Finding explicit exact and numerical solutions of nonlinear equations efficiently is of major importance and has widespread applications in numerical methods and applied mathematics. In this study, we will implement the variational homotopy perturbation method (VHPM) to find exact solution and approximate solutions to the Lienard equation for a given nonlinearity.

The VHPM provides the solution in a rapid convergent series which may lead the solution in a closed form. It is worth mentioning that the VHPM is applied with out any discretization, restrictive assumption, or transformation and is free from round-off errors. Also the VHPM provides an analytical solution by using the initial conditions only and the boundary conditions can be used only to justify the obtained result. Numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy [7].

## 2. Variational Homotopy Perturbation Method

To convey the basic idea of the variational homotopy perturbation method, we consider the following general differential equation:

$$Lu + Nu = g(x), \quad (3)$$

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where  $L$  is a linear operator,  $N$  is a nonlinear operator and  $g(x)$  is an inhomogeneous term. According to variational iteration method [4-6, 8-12], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau) \{L u_n + N \tilde{u}_n - g(\tau)\} d\tau, \quad (4)$$

where  $\lambda(\tau)$  is a Lagrange multiplier [4-6, 8-12] which can be identified optimally via the variational iteration method. The subscripts  $n$  denote the  $n$ th approximation,  $\tilde{u}_n$  is considered as a restricted variation. That is,  $\delta \tilde{u}_n = 0$  and (4) is called a correct functional. Now, we apply the homotopy perturbation method;

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p \int_0^x \lambda(\tau) \left\{ N \left( \sum_{i=0}^{\infty} p^i \tilde{u}_i \right) \right\} d\tau - p \int_0^x \lambda(\tau) g(\tau) d\tau, \quad (5)$$

which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian's polynomials. The embedding parameter  $p \in [0, 1]$  can be considered as an expanding parameter [13-18]. The homotopy perturbation method uses the homotopy parameter  $p$  as an expanding parameter [13-18] to obtain:

$$f = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \dots \quad (6)$$

If  $p \rightarrow 1$ , then (6) becomes the approximate solution of the form:

$$u = \lim_{p \rightarrow 1} f = u_0 + u_1 + u_2 + \dots \quad (7)$$

A comparison of like powers of  $p$  gives solutions of various orders.

### 3. VHPM for Lienard's equation

In this section, we consider Lienard equation (2) with initial conditions:

$$x(0) = C_1, x'(0) = C_2, \quad (8)$$

by using of initial conditions (8), we choose:

$$x_0(t) = C_1 + C_2 t, \quad (9)$$

Where (9) is an initial approximation of Eq. (2). For solving Eq. (2) via VHPM, we consider:

$$L(x) = x'', \quad (10)$$

$$N(x) = lx + mx^3 + nx^5, \quad (11)$$

where  $L$  is a linear and  $N$  is a nonlinear operator. According to the variational iteration method [4-6, 8-12], we can construct a correct functional as follows:

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\tau) \left\{ x_{n\tau\tau} + l \tilde{x}_n + m \tilde{x}_n^3 + n \tilde{x}_n^5 \right\} d\tau, \quad (12)$$

where  $\tilde{x}_n$  is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as  $\lambda = \tau - t$ , which yields the following iteration formula:

$$x_{n+1}(t) = x_n(t) + \int_0^t (\tau - t) \left\{ x_{n\tau\tau} + l x_n + m x_n^3 + n x_n^5 \right\} d\tau. \quad (13)$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} x_0 + p x_1 + p^2 x_2 + \dots &= C_1 + C_2 t \\ &+ p \int_0^t (\tau - t) l (x_0 + p x_1 + p^2 x_2 + p^3 x_3 + \dots) d\tau \\ &+ p \int_0^t (\tau - t) m (x_0 + p x_1 + p^2 x_2 + p^3 x_3 + \dots)^3 d\tau \end{aligned}$$