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Symmetry Reductions, Exact Solutions and Conservation Laws of the (3 + 1)-dimensional Nonlinear Evolution Equation

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Abstract. Using the modified CK's direct method, we build the relationship between new solutions and old ones and the (3+1)-dimensional nonlinear evolution equation. Based on the invariant group theory, Lie symmetries of the (3+1)-dimensional nonlinear evolution equation are obtained. In addition, applying the given Lie symmetry, we obtain the similarity reduction and new exact solutions, At last, we give the conservation laws of the (3+1)-dimensional nonlinear evolution equation.

Keywords: (3+1)-dimensional nonlinear evolution equation; modified CK direct method; direct symmetry method; exact solution; conservation laws

1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in several aspects of physics as well as other natural and applied sciences. One of the most important tasks in the study of NPDEs is to construct exact solutions. In order to find the exact solutions of NPDEs, the study of symmetry is very important in the area, especially in integrable system for the existence of symmetries in infinity. However, it is simple to obtain symmetry groups by using the modified CK's direct method [1−3]. And one can build the relationship between new solutions and old ones to a lot of nonlinear partial differential equations.

We consider the following (3+1)-dimensional nonlinear evolution equation in this paper

$$3u_{xz}-(2u_t+u_{xxx}-auu_x-buu_y)_y+c(u_x\partial_x^{-1}u_y)_x=0, \tag{1}$$

Where a,b and c are constants, where $a+b+c\neq 0$. For $a=2, b=0$ and $c=2$, Eq.(1) reduces to the model solved in[4,5]. In Ref.[4], the (3+1)-dimensional nonlinear evolution equation is decomposed into systems of solvable ordinary differential equations with the help of the(1+1)-dimensional AKNS equation. In Ref.[5], the perturbation technique and the Wronskian determinant of solutions were used to determine the N-soluton solutions for the reduced equation. In Ref.[6], the Hirota's bilinear method is applied to determine the necessary conditions for the complete integrability of this equation. Multiple soliton solutions are established to confirm the compatibility structure. Multiple singular soliton solutions are also derived.

The paper is organized as follows, In section 2, for $c=0$, the relationship between new solutions and old ones to the Eq. (1)is obtained by using modified CK direct method. In section 3, using the given Lie symmetry, we obtain the similarity reduction and new exact solutions. In section 4, using the given symmetry, we also get the corresponding conservation laws of the (3 +1)-dimensional nonlinear evolution equation.

2. Symmetry Groups for the nonlinear evolution equation

When $c=0$, a and b are arbitrary constants, the equation (1) becomes

$$3u_{xz}-(2u_t+u_{xxx}-auu_x-buu_y)_y=0, \tag{2}$$

Now we look for the following symmetry groups for Eq.(2),

$$u=W(x,y,z,t,U(\xi,\phi,\mu,\tau)),$$

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However, it can be proved that seeking the symmetry groups in a simple form

$$u=\alpha+\beta U(\xi,\phi,\mu,\tau), \tag{3}$$

is sufficient.

where $\alpha=\alpha(x,y,z,t),\beta=\beta(x,y,z,t),\xi=\xi(x,y,z,t),\phi=\phi(x,y,z,t)$, $\mu=\mu(x,y,z,t)$ and $\tau=\tau(x,y,z,t)$ are functions to be determined by requiring $U(\xi,\phi,\mu,\tau)$ to satisfy $u=u(x,y,z,t)$ under the transformation

$$\{u,x,y,z,t\}\to\{U,\xi,\phi,\mu,\tau\}.$$

Substituting Eq.(3) into Eq.(2) and restrict $\{U,\xi,\phi,\mu,\tau\}$ to satisfy

$$3U_{\xi\mu}-(2U_{\tau}+U_{\xi\xi\xi}-aUU_{\xi}-bUU_{\phi})_{\phi}=0, \tag{5}$$

Let the coefficients of the polynomial be zero, we have a set of differential equations with respect to U . Solving these equations, one can reach

$$\xi=c_1x+F_1(t),\phi=c_1y+\frac{bF_1(t)}{a},\mu=c_1{}^3z+c_3,\tau=c_1{}^3t+c_2,\alpha=\frac{2F_{1t}}{ac_1},\beta=c_1{}^2, \tag{6}$$

where c_1,c_2,c_3 are arbitrary constants, $F_1(t)$ is an arbitrary function of t.

For the general Lie point symmetry group of Eq.(2), we have the following theorem

Theorem 1. If $U(x,y,z,t)$ is the solution to Eq.(2), then so does $u(x,y,z,t)$, which is expressed by

$$u(x,y,z,t)=\frac{2F_{1t}}{ac_1}+c_1{}^2U(\xi,\phi,\mu,\tau), \tag{7}$$

The function ξ,ϕ,μ and τ are determined by Eq.(6).

Applying Theorem 1, we can get new exact solutions for Eq.(2) from the new kno- wn solutions. To see the relation between the symmetry groups expressed by Theorem 1 and Lie point symmetry group obtained by the standard Lie group approach, we set

$$c_1=1+\varepsilon C_1,c_2=\varepsilon C_2,c_3=\varepsilon C_3,F_1(t)=f_1\varepsilon(t), \tag{8}$$

where ε is an infinitesimal parameter, $f_1(t)$ is an arbitrary function of t. Then Eq.(7) can be written as

$$u=U+\varepsilon\delta(U), \tag{9}$$

where

$$\delta=(C_1x+f_1(t))u_x+(C_1y+\frac{bf_1(t)}{a})u_y+(3C_1z+C_3)u_z+(3C_1t+C_2)u_t+2C_1u+\frac{2f_{1t}}{a}. \tag{10}$$

As $U(x,y,z,t)$ is a solution of the Eq.(2) under the condition of Theorem 1, we can replace U with u in Eq.(10) to get the corresponding Lie symmetry of the Eq. (2), and then the equivalent vector expression of the symmetry is

$$V=(C_1x+f_1(t))\frac{\partial}{\partial x}+(C_1y+\frac{bf_1(t)}{a})\frac{\partial}{\partial y}+(3C_1z+C_3)\frac{\partial}{\partial z}+(3C_1t+C_2)\frac{\partial}{\partial t}-(2C_1u+\frac{2f_{1t}}{a})\frac{\partial}{\partial u},$$

which is exactly the same as that obtained by the standard Lie group approach [7].

3. Similarity Reduction and New Exact Solution for nonlinear Evolution Equation

In this section, we will discuss the reduction and solutions of Eq. (2)for the cases of Theorem. First, we look for invariants ξ_1,η_1,τ_1 and $\theta(\xi_1,\eta_1,\tau_1)$. To do this, one must solve the following characteristic equations,

$$\frac{dx}{C_1x+f_1(t)}=\frac{dy}{C_1y+\dfrac{bf_1(t)}{a}}=\frac{dz}{3C_1z+C_3}=\frac{dt}{3C_1t+C_2}=\frac{du}{-(2C_1u+\dfrac{2f_{1t}}{a})}, \tag{11}$$

Solving Eq.(11), and one can obtain

$$u=\frac{\theta(\xi_1,\eta_1,\tau_1)-\int\dfrac{2f_{1t}}{a(3c_1t+c_2)^{1/3}}dt}{(3c_1t+c_2)^{2/3}}, \tag{12}$$

where θ is an arbitrary function of the corresponding variables and

$$\xi_1=\frac{x}{(3c_1t+c_2)^{1/3}}-\int\frac{f_1}{(3c_1t+c_2)^{4/3}}dt,\eta_1=\frac{y}{(3c_1t+c_2)^{1/3}}-\int\frac{bf_1}{a(3c_1t+c_2)^{4/3}}dt,\tau_1=\frac{z+\frac{c_3}{3c_1}}{3c_1t+c_2},$$

By substituting Eq.(12) into Eq.(2), we obtain the reduction of Eq.(2),

$$a\theta\theta_{\xi_1\eta_1}+b\theta\theta_{\eta_1\eta_1}+a\theta_{\xi_1}\theta_{\eta_1}+b\theta_{\eta_1}{}^2+6c_1\theta_{\eta_1}+2c_1\xi_1\theta_{\xi_1\eta_1}+2c_1\eta_1\theta_{\eta_1\eta_1}-\theta_{\xi_1\xi_1\xi_1\eta_1}+3\theta_{\xi_1\tau_1}+6c_1\tau_1\theta_{\eta_1\tau_1}=0, \quad (13)$$

where $\theta=\theta(\xi_1,\eta_1,\tau_1)$. In order to search for the explicit exact solutions, we will apply two methods [8,9] to solve Eq. (13).

Case1. when $c_1=0$, the equation (13) becomes

$$a\theta\theta_{\xi_1\eta_1}+b\theta\theta_{\eta_1\eta_1}+a\theta_{\xi_1}\theta_{\eta_1}+b\theta_{\eta_1}{}^2-\theta_{\xi_1\xi_1\xi_1\eta_1}+3\theta_{\xi_1\tau_1}=0, \quad (14)$$

Using a new expansion method [8] to solve Eq. (14) , we find the following solutions of the Eq.(2),

$$u_1=c_2^{-2/3}\left(\frac{a_2(8k-3l)}{12k}+a_2\coth(\xi_1+k\eta_1+l\tau_1)^2-2ac_2^{-1/3}\int f_{1t}dt\right),$$

$$u_2=c_2^{-2/3}\left(\frac{a_{-2}}{\coth(\xi_1+k\eta_1+\frac{4k(2a_{-2}-3a_0)}{3a_{-2}}\tau_1)^2}+a_0+a_{-2}\coth(\xi_1+k\eta_1+\frac{4k(2a_{-2}-3a_0)}{3a_{-2}}\tau_1)^2\right.$$
$$\left.-2ac_2^{-1/3}\int f_{1t}dt\right),$$

$$u_{3,4}=c_2^{-2/3}\left(a_0\pm b_2\coth(\xi_1+k\eta_1+\frac{k(5b_2+6a_0)}{3b_2}\tau_1)^2+b_2\coth(\xi_1+k\eta_1+\frac{k(5b_2+6a_0)}{3b_2}\tau_1)\right.$$
$$\left.\operatorname{csch}(\xi_1+k\eta_1+\frac{k(5b_2+6a_0)}{3b_2}\tau_1)^2-2ac_2^{-1/3}\int f_{1t}dt\right),$$

$$u_{5,6}=c_2^{-2/3}\left(\pm\frac{c_{-2}}{\coth(\xi_1+k\eta_1-\frac{k(-5b_2+6a_0)}{c_{-2}}\tau_1)^2}+a_0-\frac{c_{-2}\operatorname{csch}(\xi_1+k\eta_1-\frac{k(-5b_2+6a_0)}{c_{-2}}\tau_1)}{\coth(\xi_1+k\eta_1-\frac{k(-5b_2+6a_0)}{c_{-2}}\tau_1)^2}\right.$$
$$\left.-2ac_2^{-1/3}\int f_{1t}dt\right),$$

$$u_7=c_2^{-2/3}\left(-\frac{a_2(8k+3l)}{12k}+a_2\cot(\xi_1+k\eta_1+l\tau_1)^2-2ac_2^{-1/3}\int f_{1t}dt\right),$$

$$u_8=c_2^{-2/3}\left(\frac{a_{-2}}{\cot(\xi_1+k\eta_1+\frac{4k(2a_{-2}-3a_0)}{3a_{-2}}\tau_1)^2}+a_0+a_{-2}\cot(\xi_1+k\eta_1+\frac{4k(2a_{-2}-3a_0)}{3a_{-2}}\tau_1)^2\right.$$
$$\left.-2ac_2^{-1/3}\int f_{1t}dt\right),$$

$$u_{9,10}=c_2^{-2/3}\left(a_0\pm b_2\cot(\xi_1+k\eta_1+\frac{k(5b_2+6a_0)}{3b_2}\tau_1)^2+b_2\cot(\xi_1+k\eta_1+\frac{k(5b_2+6a_0)}{3b_2}\tau_1)\right.$$
$$\left.\csc(\xi_1+k\eta_1+\frac{k(5b_2+6a_0)}{3b_2}\tau_1)^2-2ac_2^{-1/3}\int f_{1t}dt\right),$$

$$u_{11,12}=c_2^{-2/3}\left(\frac{Ic_{-2}}{\cot(\xi_1+k\eta_1-\frac{k(-5b_2+6Ia_0)}{c_{-2}}\tau_1)^2}+a_0-\frac{c_{-2}\csc(\xi_1+k\eta_1-\frac{k(-5b_2+6Ia_0)}{c_{-2}}\tau_1)}{\cot(\xi_1+k\eta_1-\frac{k(-5b_2+6Ia_0)}{c_{-2}}\tau_1)^2}\right.$$
$$\left.-2ac_2^{-1/3}\int f_{1t}dt\right),$$

where $\xi_1 = c_2^{-1/3}x - c_2^{-4/3}\int f_1 dt, \eta_1 = c_2^{-1/3}y - abc_2^{-4/3}\int f_1 dt, \tau_1 = \frac{z}{c_2}, \ I^2 = -1$ and $a, b, c_2, b_2, c_{-2}, a_0, a_{-2}, k, l$ are arbitrary constants.

Case2 . In this case $c_1 \neq 0,$ we will apply a direct symmetry method [9] to get the reduction of the Eq. (13). We can get

$$\theta = -\frac{2c_1\xi_1}{a} + \varpi(\xi_2, \tau_2), \tag{19}$$

where ϖ is arbitrary function of the corresponding variables and

$$\xi_2 = \eta_1 - \frac{b\xi_1}{a}, \tau_2 = \tau_1, \tag{20}$$

By substituting Eq. (19) and Eq.(20) into Eq.(13), we obtain the reduction of Eq.(13),

$$(6c_1\tau_2 - \frac{3b}{a})\varpi_{\xi_2\tau_2} + 4c_1\varpi_{\xi_2} + \frac{b^3}{a^3}\varpi_{\xi_2\xi_2\xi_2\xi_2} + 2c_1\xi_2\varpi_{\xi_2\xi_2} = 0. \tag{21}$$

We can easily get the general solution of Eq.(21) by using Maple software

$$\begin{aligned}\varpi = F(\tau_2) + \int c_4(2c_1a\tau_2 - b)^{-(c_2/6c_1a^3+2/3)}&(C1hypergeom([-\frac{c_2}{6c_1a^3}],[\frac{1}{3},\frac{2}{3}],-\frac{2c_1a^3\xi_2^{\ 3}}{9b^3}) + C2\\ &hypergeom([\frac{1}{3}-\frac{c_2}{6c_1a^3}],[\frac{2}{3},\frac{4}{3}],-\frac{2c_1a^3\xi_2^{\ 3}}{9b^3})\xi_2 + C3hypergeom([\frac{2}{3}-\frac{c_2}{6c_1a^3}],[\frac{4}{3},\frac{5}{3}],\\ &-\frac{2c_1a^3\xi_2^{\ 3}}{9b^3}))d\xi_2,\end{aligned} \tag{22}$$

where $F(\tau_2)$ is an arbitrary function. Therefore, we can obtain a new solution to the Eq. (2) by substituting Eq. (22) into Eq. (19)with the help of Eq. (12),

$$\begin{aligned}u_{13} = &(-\frac{2c_1}{a}(\frac{x}{(3c_1t+c_2)^{1/3}} - \int\frac{f_1}{(3c_1t+c_2)^{4/3}}dt) - \int\frac{2f_{1t}}{a(3c_1t+c_2)^{1/3}}dt + F(\frac{z+\frac{c_3}{3c_1}}{3c_1t+c_2})\\ &+\int c_4(\frac{2c_1az+2ac_3/3}{3c_1t+c_2} - b)^{-(c_2/6c_1a^3+2/3)}(C1hypergeom([-\frac{c_2}{6c_1a^3}],[\frac{1}{3},\frac{2}{3}],-\frac{2c_1a^3(ay-bx)^3}{9b^3(3c_1t+c_2)})\\ &+C2hypergeom([\frac{1}{3}-\frac{c_2}{6c_1a^3}],[\frac{2}{3},\frac{4}{3}],-\frac{2c_1a^3(ay-bx)^3}{9b^3(3c_1t+c_2)})\frac{ay-bx}{(3c_1t+c_2)^{1/3}}\\ &+C3hypergeom([\frac{2}{3}-\frac{c_2}{6c_1a^3}],[\frac{4}{3},\frac{5}{3}],-\frac{2c_1a^3(ay-bx)^3}{9b^3(3c_1t+c_2)}))d(\frac{ay-bx}{(3c_1t+c_2)^{1/3}}))/(3c_1t+c_2)^{2/3},\end{aligned}$$

where f_1 is an arbitrary function of t.

4. conservation laws

Next , we study the conservation laws by using the adjoint equation and symmetries of the Eq. (2). For Eq. (2), the adjoint equation has the form

$$au_{xy}v - av_xu_y - 2bv_yu_y + auv_{xy} + buv_{yy} - 2v_{yt} + 3v_{xz} - v_{xxxy} = 0, \tag{23}$$

and the Lagrangian is in the symmetrized form

$$L = v(3u_{xz} - (2u_t + u_{xxx} - auu_x - buu_y)_y). \tag{24}$$

Theorem 2. Every Lie point, Lie–Boaklund and non-local symmetry of Eq.(2) provides a conservation law for Eq. (2)and the adjoint equation [10] . Then the elements of conservation vector (C^1, C^2, C^3, C^4) are defined by the following expression

$$C^i=\xi^i L+W^\alpha[\frac{\partial L}{\partial u_i}-D_j(\frac{\partial L}{\partial u_{ij}})+D_jD_k(\frac{\partial L}{\partial u_{ijk}})-D_jD_kD_r(\frac{\partial L}{\partial u_{ijkr}})]+D_j(W)[\frac{\partial L}{\partial u_{ij}}-D_k(\frac{\partial L}{\partial u_{ijk}})$$
$$+D_kD_r(\frac{\partial L}{\partial u_{ijkr}})]+D_jD_k(W)[\frac{\partial L}{\partial u_{ijk}}-D_r(\frac{\partial L}{\partial u_{ijkr}})]+D_jD_kD_r(W)\frac{\partial L}{\partial u_{ijkr}}, \quad (25)$$

where $W^\alpha=\eta-\xi^j u_j$. The conserved vector corresponding to an operator is

$$V=\xi^1(x,y,z,t,u)\frac{\partial}{\partial t}+\xi^2(x,y,z,t,u)\frac{\partial}{\partial x}+\xi^3(x,y,z,t,u)\frac{\partial}{\partial y}+\xi^4(x,y,z,t,u)\frac{\partial}{\partial z}+\eta(x,y,z,t,u)\frac{\partial}{\partial u}.$$

The operator V yields the conservation law $D_t(C^1)+D_x(C^2)+D_y(C^3)+D_z(C^4)$ $=0$, where the conserved vector $C=(C^1,C^2,C^3,C^4)$ is given by Eq.(25) and has the components

$$C^1=\xi^1 L+W^1 v_y-2vD_yW^1, \quad (26)$$
$$C^2=\xi^2 L-W^1(auv_y+3v_z+v_{xxy})+(auv-v_{xy})D_yW^1-v_yD_{xy}W^1+vD_{xxy}W^1, \quad (27)$$
$$C^3=\xi^3 L-W^1((2-b)u_yv-buv_y-au_xv-auv_x+2v_t+v_{xxx})-2vD_tW^1+(auv-v_{xx})D_xW^1$$
$$+buvD_yW^1+v_xD_{xx}W^1-D_{xxx}W^1, \quad (28)$$
$$C^4=\xi^4 L-3W^1v_x+3vD_xW^1, \quad (29)$$

Thus, Eqs. (26), (27), (28) and (29) define the corresponding components of a nonlocal conservation laws for the system of Eq.(2) and (23) corresponding to any operator V admitted by Eq.(2).

Let us make more detailed calculations for the operator

$$V=3t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+3z\frac{\partial}{\partial z}-2u\frac{\partial}{\partial u},$$

For this operator, we have

$$W^1=-2u-(3tu_t+xu_x+yu_y+3zu_z),$$

and Eq. (25) written for the Lagrangian (32) yields the conserved vector

$$C^1=3tv(3u_{xz}-(2u_t+u_{xxx}-auu_x-buu_y)_y)-v_y(2u+(3tu_t+xu_x+yu_y+3zu_z))$$
$$+2v(2u_y+(3tu_{ty}+xu_{xy}+u_y+yu_{yy}+3zu_{zy})),$$
$$C^2=xv(3u_{xz}-(2u_t+u_{xxx}-auu_x-buu_y)_y)+(2u+(3tu_t+xu_x+yu_y+3zu_z))(auv_y+3v_z$$
$$+v_{xxy})+(auv-v_{xy})(2u_y+(3tu_{ty}+xu_{xy}+u_y+yu_{yy}+3zu_{zy}))-v_y(2u_{xy}+(3tu_{xty}+u_{xy}+xu_{xxy}$$
$$+u_{xy}+yu_{xyy}+3zu_{xzy}))+v(2u_{xxy}+(3tu_{xxyt}+2u_{xxy}+xu_{xxxy}+u_{xxy}+yu_{xxyy}+3zu_{xxyz})),$$
$$C^3=yv(3u_{xz}-(2u_t+u_{xxx}-auu_x-buu_y)_y)+(2u+(3tu_t+xu_x+yu_y+3zu_z))((2-b)u_yv$$
$$-buv_y-au_xv-auv_x+2v_t+v_{xxx})+2v(2u_t+(3u_t+3tu_{tt}+xu_{xt}+yu_{yt}+3zu_{zt}))-(auv$$
$$-v_{xx})(2u_x+(3tu_{xt}+u_x+xu_{xx}+yu_{xy}+3zu_{xz}))-buv(2u_y+(3tu_{ty}+xu_{xy}+u_y+yu_{yy}$$
$$+3zu_{zy}))-v_x(2u_{xx}+(3tu_{xxt}+2u_{xx}+xu_{xxx}+yu_{xxy}+3zu_{xxz}))+(2u_{xx}+(3tu_{xxxt}$$
$$+3u_{xxx}+xu_{xxxx}+yu_{xxxy}+3zu_{xxxz})),$$
$$C^4=3zv(3u_{xz}-(2u_t+u_{xxx}-auu_x-buu_y)_y)+3(2u+(3tu_t+xu_x+yu_y+3zu_z))v_x$$
$$-3v(2u_x+(3tu_{xt}+u_x+xu_{xx}+yu_{xy}+3zu_{xz})),$$

This vector involves an arbitrary solution v of the adjoint Eq.(31) and provides an infinite number of the conservation laws.

Remark. With the aid of Maple, we have checked that the above vector (C^1,C^2,C^3,C^4) is the conservation vector of Eq.(2).

5. Conclusions

In this paper, by using the modified CK's direct method, we derive the relationship between the new solutions and the old ones some new exact solutions of Eq. (2). Symmetry groups and Lie point symmetry group for Eq.(2) are also obtained. Based on the given Lie symmetry, we reduce the Eq. (2)to (1+1)-dimensional PDE and get new exact solution. Lastly , we give the conservation laws of Eq. (2).

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