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A Variational Principle For Nonlinear Local Pressure

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Abstract. In this paper, we introduce a concept of nonlinear local topological pressure defined via open covers and establish a corresponding variational principle. Furthermore, we provide multiple equivalent characterizations of nonlinear pressure using different cover-based approaches.

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Key words: Local variational principle, Nonlinear topological pressure, Nonlinear local pressure, Local entropy.

1 Introduction and main results

1.1 Nonlinear topological pressure

Topological pressure is a fundamental invariant in dynamical systems theory, serving as a natural generalization of topological entropy. The concept was first introduced by Ruelle [1] in the context of expansive systems and later extended by Walters [2] to more general settings. A *topological dynamical system* (TDS for short) is a pair (X,T) consisting of a compact metric space X and a surjective continuous map $T: X \to X$. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X, $\mathcal{M}(X,T)$ be the set of T-invariant Borel probability measures on X and $\mathcal{M}^e(X,T)$ be the set of ergodic T-invariant measures on X. Given a continuous function $f: X \to \mathbb{R}$, the topological pressure $P(T,f)$ satisfies the following variational principle:

$$P(T,f) = \sup_{\mu \in \mathcal{M}(X,T)} \left\{ h_\mu(T) + \int_X f(x) d\mu(x) \right\},$$

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where $h_\mu(T)$ is the measure-theoretic entropy of μ. Topological pressure and its variational principle form the foundation of thermodynamic formalism and play a fundamental role in the dimension theory of dynamical systems.

Recently, Buzzi, Kloeckner and Leplaideur [3] developed a nonlinear thermodynamic formalism based on the Curie-Weiss mean-field theory [4]. By transforming the statistical mechanics of generalized mean-field models into dynamical systems theory, they established a variational principle for nonlinear topological pressures, provided that the system possesses an abundance of ergodic measures. Subsequently, Barreira and Holanda [5, 6] extended this framework by introducing a higher-dimensional generalization of the nonlinear thermodynamic formalism and its continuous-time counterpart for flows, respectively. Kucherenko [7] established a connection between the nonlinear thermodynamic formalism and the theory of generalized rotation sets. Yang, Chen and Zhou [8] introduced the notion of nonlinear weighted topological pressure for factor maps and established an associated variational principle. Ding and Wang [9] introduced the nonlinear topological pressure for subsets and established corresponding variational principles.

Now we recall the background and the main result of [3]. We call a function $\mathcal{E}:\mathcal{M}(X)\to\mathbb{R}$ is an *energy* if it is continuous in the weak-star topology. For instance, given continuous functions $f:X\to\mathbb{R}$ and $F:\mathbb{R}\to\mathbb{R}$, the function $\mathcal{E}$ defined by

$$\mathcal{E}(\mu)=F\left(\int f d\mu\right)$$

is then an energy on $\mathcal{M}(X)$. For a given TDS (X,T) and an energy $\mathcal{E}:\mathcal{M}(X)\to\mathbb{R}$, the *nonlinear topological pressure* $P(T,\mathcal{E})$ is defined as

$$P(T,\mathcal{E})=\lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(T,\mathcal{E},\varepsilon),$$

where

$$P_n(T,\mathcal{E},\varepsilon)=\sup\left\{\sum_{x\in E}e^{n\mathcal{E}(\triangle_x^n)}: E\, is\, an\, (n,\varepsilon)\text{-}separated\, set\, of\, X\right\},$$

and $\triangle_x^n:=\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^ix}$. Assuming that $(T,\mathcal{E})$ has an abundance of ergodic measures, they proved that

$$P(T,\mathcal{E})=\sup_{\mu\in\mathcal{M}(X,T)}\left\{h_\mu(T)+\mathcal{E}(\mu)\right\}.$$

1.2 Local entropy and pressure

The local theory of entropy and pressure is a foundational framework in dynamical systems, with profound connections to various concepts including entropy pairs, entropy sets, entropy points, and entropy structures, etc. Given a TDS (X,T) and an open cover $\mathcal{U}$ of X, Romagnoli [10] introduced two types of measure-theoretic entropy relative to $\mathcal{U}$: