

A Variational Principle For Nonlinear Local Pressure

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Abstract. In this paper, we introduce a concept of nonlinear local topological pressure defined via open covers and establish a corresponding variational principle. Furthermore, we provide multiple equivalent characterizations of nonlinear pressure using different cover-based approaches.

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1 Introduction and main results

1.1 Nonlinear topological pressure

Topological pressure is a fundamental invariant in dynamical systems theory, serving as a natural generalization of topological entropy. The concept was first introduced by Ruelle [1] in the context of expansive systems and later extended by Walters [2] to more general settings. A *topological dynamical system* (TDS for short) is a pair (X, T) consisting of a compact metric space X and a surjective continuous map $T : X \rightarrow X$. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X , $\mathcal{M}(X, T)$ be the set of T -invariant Borel probability measures on X and $\mathcal{M}^e(X, T)$ be the set of ergodic T -invariant measures on X . Given a continuous function $f : X \rightarrow \mathbb{R}$, the topological pressure $P(T, f)$ satisfies the following variational principle:

$$P(T, f) = \sup_{\mu \in \mathcal{M}(X, T)} \left\{ h_{\mu}(T) + \int_X f(x) d\mu(x) \right\},$$

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where $h_\mu(T)$ is the measure-theoretic entropy of μ . Topological pressure and its variational principle form the foundation of thermodynamic formalism and play a fundamental role in the dimension theory of dynamical systems.

Recently, Buzzi, Kloeckner and Leplaideur [3] developed a nonlinear thermodynamic formalism based on the Curie-Weiss mean-field theory [4]. By transforming the statistical mechanics of generalized mean-field models into dynamical systems theory, they established a variational principle for nonlinear topological pressures, provided that the system possesses an abundance of ergodic measures. Subsequently, Barreira and Holanda [5, 6] extended this framework by introducing a higher-dimensional generalization of the nonlinear thermodynamic formalism and its continuous-time counterpart for flows, respectively. Kucherenko [7] established a connection between the nonlinear thermodynamic formalism and the theory of generalized rotation sets. Yang, Chen and Zhou [8] introduced the notion of nonlinear weighted topological pressure for factor maps and established an associated variational principle. Ding and Wang [9] introduced the nonlinear topological pressure for subsets and established corresponding variational principles.

Now we recall the background and the main result of [3]. We call a function $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ is an *energy* if it is continuous in the weak-star topology. For instance, given continuous functions $f : X \rightarrow \mathbb{R}$ and $F : \mathbb{R} \rightarrow \mathbb{R}$, the function \mathcal{E} defined by

$$\mathcal{E}(\mu) = F\left(\int f d\mu\right)$$

is then an energy on $\mathcal{M}(X)$. For a given TDS (X, T) and an energy $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$, the *nonlinear topological pressure* $P(T, \mathcal{E})$ is defined as

$$P(T, \mathcal{E}) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(T, \mathcal{E}, \epsilon),$$

where

$$P_n(T, \mathcal{E}, \epsilon) = \sup \left\{ \sum_{x \in E} e^{n\mathcal{E}(\Delta_x^n)} : E \text{ is an } (n, \epsilon)\text{-separated set of } X \right\},$$

and $\Delta_x^n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x}$. Assuming that (T, \mathcal{E}) has an abundance of ergodic measures, they proved that

$$P(T, \mathcal{E}) = \sup_{\mu \in \mathcal{M}(X, T)} \{h_\mu(T) + \mathcal{E}(\mu)\}.$$

1.2 Local entropy and pressure

The local theory of entropy and pressure is a foundational framework in dynamical systems, with profound connections to various concepts including entropy pairs, entropy sets, entropy points, and entropy structures, etc. Given a TDS (X, T) and an open cover \mathcal{U} of X , Romagnoli [10] introduced two types of measure-theoretic entropy relative to \mathcal{U} :

$h_\mu(T, \mathcal{U})$ and $h_\mu^+(T, \mathcal{U})$ (see Section 2 for precise definitions). He established the following variational principle for local entropy:

$$h_{top}(T, \mathcal{U}) = \sup_{\mu \in \mathcal{M}(X, T)} h_\mu(T, \mathcal{U}).$$

For invertible systems (X, T) , Glasner and Weiss [11] proved that

$$h_{top}(T, \mathcal{U}) = \sup_{\mu \in \mathcal{M}(X, T)} h_\mu^+(T, \mathcal{U}).$$

Further developments in local thermodynamic formalism were made by Huang, Ye, and Zhang [12], who derived a relative local variational principle for entropy. Building on these results Huang and Yi [13] extended the variational principle for local entropy to the case of pressure. Specifically, for a continuous potential function $f: X \rightarrow \mathbb{R}$, they showed the local pressure $P(T, f; \mathcal{U})$ satisfies

$$P(T, f; \mathcal{U}) = \sup_{\mu \in \mathcal{M}(X, T)} \left\{ h_\mu(T, \mathcal{U}) + \int_X f(x) d\mu(x) \right\}.$$

In recent years, Wu [14] investigated several notions of local pressure for subsets and measures, defined using the Carathéodory-Pesin construction. Cai [15] established the variational principle for weighted local pressure. The study of local conditional pressures for additive or sub-additive potentials was developed by several authors (see, e.g., [16–18]). In the relative setting, local topological pressures were investigated in [19–21]. The case of amenable group actions was studied in [22–24].

1.3 Main result

In this paper, we introduce the notion of nonlinear local topological pressure and prove a corresponding variational principle.

Let (X, T) be a TDS. We define a *cover* of X as a finite collection of Borel subsets whose union equals X , and a *partition* of X as a cover with pairwise disjoint elements. Denote by $\mathcal{C}_X, \mathcal{P}_X$ and \mathcal{C}_X^o the set of *covers*, *partitions* and *open covers* of X , respectively. Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, if each element of \mathcal{U} is contained in some element of \mathcal{V} , then we say \mathcal{U} is *finer* than \mathcal{V} (denoted by $\mathcal{U} \succeq \mathcal{V}$). Denote $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and $\mathcal{U}_0^{n-1} = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}$.

Recall that $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ is called an *energy* if it is a continuous function on $\mathcal{M}(X)$.

Definition 1.1. Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^o$. Let

$$p_n(T, \mathcal{E}; \mathcal{U}) = \inf \left\{ \sum_{V \in \mathcal{V}} \sup_{x \in V} e^{n\mathcal{E}(\Delta_x^n)} : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}_0^{n-1} \right\}, \quad (1.1)$$

where $\Delta_x^n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x}$. We define the *nonlinear local pressure* of $(T, \mathcal{E}; \mathcal{U})$ by:

$$P(T, \mathcal{E}; \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log p_n(T, \mathcal{E}; \mathcal{U}), \quad (1.2)$$

and define the *nonlinear lower local pressure* of $(T, \mathcal{E}; \mathcal{U})$ by:

$$\underline{P}(T, \mathcal{E}; \mathcal{U}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log p_n(T, \mathcal{E}; \mathcal{U}). \quad (1.3)$$

Note that $\log p_n(T, \mathcal{E}; \mathcal{U})$ is generally not sub-additive. As a result, the sequence $\{\frac{1}{n} \log p_n(T, \mathcal{E}; \mathcal{U})\}_{n \in \mathbb{N}}$ may fail to converge. However, under the following abundance of ergodic measures condition, we show in Theorem 1.1 that the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \log p_n(T, \mathcal{E}; \mathcal{U})$ exists.

Definition 1.2. We say that $(T, \mathcal{E}; \mathcal{U})$ has an *abundance of ergodic measures*, if for any $\mu \in \mathcal{M}(X, T)$ and $\varepsilon > 0$, there is an ergodic measure $\nu \in \mathcal{M}^e(X, T)$ such that $h_\nu(T, \mathcal{U}) + \mathcal{E}(\nu) > h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu) - \varepsilon$.

Note that the ergodic measures of uniformly hyperbolic systems or systems with specification are entropy-dense, i.e., the invariant measures can be approached by ergodic measures both in the weak star topology and in entropy (see [25, 26]). Therefore, if (Σ, σ) is a full shift, $\mathcal{E}: \mathcal{M}(\Sigma) \rightarrow \mathbb{R}$ is an energy and \mathcal{U} is a generator, then $(T, \mathcal{E}; \mathcal{U})$ has an abundance of ergodic measures.

Our main conclusions are as follows:

Theorem 1.1. (Local variational principle): Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^0$. Suppose that $(T, \mathcal{E}; \mathcal{U})$ has an abundance of ergodic measures. Then the nonlinear local pressure $P(T, \mathcal{E}; \mathcal{U})$ satisfies

$$P(T, \mathcal{E}; \mathcal{U}) = \underline{P}(T, \mathcal{E}; \mathcal{U}) = \sup_{\mu \in \mathcal{M}(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\}.$$

In addition to the variational principle, we also demonstrate how the nonlinear local pressure relates to the global nonlinear topological pressure:

Theorem 1.2. Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy. Then

$$P(T, \mathcal{E}) = \sup_{\mathcal{U} \in \mathcal{C}_X^0} P(T, \mathcal{E}; \mathcal{U}), \text{ and } \underline{P}(T, \mathcal{E}) = \sup_{\mathcal{U} \in \mathcal{C}_X^0} \underline{P}(T, \mathcal{E}; \mathcal{U}).$$

Theorem 1.2 is a special case of Theorems 4.1 and 4.2. In Section 4, we introduce several different definitions of nonlinear local pressure and reveal their relationship with nonlinear topological pressure.

The paper is organized as follows. Section 2 introduces the fundamental definitions and properties of local entropy and pressure. In Section 3, we prove Theorem 1.1, while Section 4 is devoted to the proof of Theorems 4.1 and 4.2, from which Theorem 1.2 follows as a direct consequence.

2 Preliminaries

2.1 Wasserstein distance on $\mathcal{M}(X)$

We equip the space of probability measures $\mathcal{M}(X)$ with the Wasserstein distance [27]. For $\mu_1, \mu_2 \in \mathcal{M}(X)$, the distance is defined as:

$$W(\mu_1, \mu_2) = \sup\{\mu_1(f) - \mu_2(f) : f : X \rightarrow \mathbb{R} \text{ is 1-Lipschitz}\}.$$

Since X is compact, the Wasserstein distance induces the weak-star topology on $\mathcal{M}(X)$, and the Wasserstein distance can be bounded by the total variation distance [3]:

$$W(\mu_1, \mu_2) \leq \text{diam}(X) \|\mu_1 - \mu_2\|_{TV},$$

where $\|\mu_1 - \mu_2\|_{TV} = \sup\{\mu_1(f) - \mu_2(f) : f : X \rightarrow \mathbb{R} \text{ is measurable and } \|f\|_\infty \leq 1\}$.

2.2 Measure-theoretic entropies

Let (X, T) be a TDS. For $\mathcal{U} \in \mathcal{C}_X$, denote $H(\mathcal{U}) = \log N(\mathcal{U})$, where $N(\mathcal{U})$ is the minimum among the cardinals of the sub-covers of \mathcal{U} . Then we define

$$h_{top}(T, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\mathcal{U}_0^{n-1}) = \inf_{n \geq 1} \frac{1}{n} H(\mathcal{U}_0^{n-1}),$$

as the *topological entropy* of \mathcal{U} . Define

$$h_{top}(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_{top}(T, \mathcal{U})$$

as the *topological entropy* of (X, T) .

Given a partition $\alpha \in \mathcal{P}_X$ and a measure $\mu \in \mathcal{M}(X, T)$, the *measure-theoretic entropy* of μ relative to α is defined as :

$$h_\mu(T, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}) = \inf_{n \geq 1} \frac{1}{n} H_\mu(\alpha_0^{n-1}),$$

where

$$H_\mu(\alpha) = \sum_{A \in \alpha} -\mu(A) \log \mu(A).$$

The *measure-theoretic entropy* of μ is then defined by:

$$h_\mu(T) = \sup_{\alpha \in \mathcal{P}_X} h_\mu(T, \alpha).$$

For a cover $\mathcal{U} \in \mathcal{C}_X$, Romagnoli [10] introduced the following two *measure-theoretic entropies relative to \mathcal{U}* :

$$h_\mu(T, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}) \quad \text{and} \quad h_\mu^+(T, \mathcal{U}) = \inf_{\alpha \succeq \mathcal{U}, \alpha \in \mathcal{P}_X} h_\mu(T, \alpha),$$

where

$$H_\mu(\mathcal{U}) = \inf_{\alpha \succeq \mathcal{U}, \alpha \in \mathcal{P}_X} H_\mu(\alpha).$$

By [10, Lemma 8], for $\mu \in \mathcal{M}(X)$ and $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $H_\mu(\mathcal{U})$ satisfies the following properties:

- (i) $0 \leq H_\mu(\mathcal{U}) \leq \log N(\mathcal{U})$;
- (ii) If $\mathcal{U} \succeq \mathcal{V}$, then $H_\mu(\mathcal{U}) \geq H_\mu(\mathcal{V})$;
- (iii) $H_\mu(\mathcal{U} \vee \mathcal{V}) \leq H_\mu(\mathcal{U}) + H_\mu(\mathcal{V})$;
- (iv) $H_\mu(T^{-1}\mathcal{U}) \leq H_{T_*\mu}(\mathcal{U})$, where $T_*\mu = \mu \circ T^{-1}$, and, the equality holds when (X, T) is invertible.

Moreover, if $\mu \in \mathcal{M}(X, T)$, it's proved by [13, Lemma 2.7] that

$$h_\mu(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu(T, \mathcal{U}).$$

Now given $\mathcal{U} = \{U_1, U_2, \dots, U_d\} \in \mathcal{C}_X^o$, denote

$$\mathcal{U}^* = \{\alpha \in \mathcal{P}_X : \alpha = \{A_1, A_2, \dots, A_d\}, A_i \subset U_i, i = 1, 2, \dots, d\},$$

where A_i could be empty for some i . By [28, Lemma 2] and [13, Corollary], the following properties hold.

Proposition 2.1. Let (X, T) be a TDS, $\mu \in \mathcal{M}(X, T)$ and $\mathcal{U} \in \mathcal{C}_X^o$. The following properties hold:

1. $h_\mu^+(T, \mathcal{U}) = \inf_{\alpha \in \mathcal{U}^*} h_\mu(T, \alpha)$;
2. If in addition (X, T) is invertible, then $h_\mu(T, \mathcal{U}) = h_\mu^+(T, \mathcal{U})$.

2.3 Some properties of nonlinear local pressures

Let (X, T) be a TDS, $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^o$. The nonlinear local pressure $P(T, \mathcal{E}; \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log p_n(T, \mathcal{E}; \mathcal{U})$ is defined as in Definition 1.1. We first provide an equivalent statement of $p_n(T, \mathcal{E}; \mathcal{U})$.

For $\mathcal{V} \in \mathcal{C}_X$, let α be the Borel partition generated by \mathcal{V} . Define

$$\mathcal{P}^*(\mathcal{V}) = \{\beta \in \mathcal{P}_X : \beta \succeq \mathcal{V} \text{ and each atom of } \beta \text{ is the union of some atoms of } \alpha\}.$$

Lemma 2.1. $\mathcal{P}^*(\mathcal{V})$ is a finite set, and for each $n \in \mathbb{N}$,

$$\inf_{\beta \in \mathcal{C}_X, \beta \succeq \mathcal{V}} \sum_{B \in \beta} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} = \min \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} : \beta \in \mathcal{P}^*(\mathcal{V}) \right\}.$$

In particular, by taking $\mathcal{V} = \mathcal{U}_0^{n-1}$, we have

$$p_n(T, \mathcal{E}; \mathcal{U}) = \min \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} : \beta \in \mathcal{P}^*(\mathcal{U}_0^{n-1}) \right\}.$$

Proof. The proof of [13, Lemma 2.1] remains valid for this lemma upon replacing $f_n(x)$ by $n\mathcal{E}(\Delta_x^n)$. We therefore omit the details of the proof. \square

Let (Y, S) and (X, T) be two TDS. We say $\pi: Y \rightarrow X$ is a *factor map* from (Y, S) to (X, T) if π is a surjective and continuous map satisfying $\pi \circ S = T \circ \pi$. In this case, (Y, S) is called an *extension* of (X, T) , and (X, T) is called a *factor* of (Y, S) . In addition, if π is also injective then it's called an *isomorphism*. A map $\pi: Y \rightarrow X$ induces a map $\pi_*: \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$ defined by $\pi_*\mu = \mu \circ \pi^{-1}$.

Lemma 2.2 ([10], Proposition 6). *Let (Y, S) and (X, T) be two TDS. If $\pi: (Y, S) \rightarrow (X, T)$ is a factor map, then for any $\nu \in \mathcal{M}_S(Y)$ and $\mathcal{U} \in \mathcal{C}_X^o$, we have $h_\nu(S, \pi^{-1}\mathcal{U}) = h_{\pi_*\nu}(T, \mathcal{U})$.*

Lemma 2.3. *Let $\pi: (Y, S) \rightarrow (X, T)$ be a factor map, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^o$. Then $P(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}) = P(T, \mathcal{E}; \mathcal{U})$.*

Proof. Fix $n \in \mathbb{N}$ and let $\mathcal{V} \in \mathcal{C}_Y$ satisfy $\mathcal{V} \succeq \mathcal{U}_0^{n-1}$. Then $\pi^{-1}\mathcal{V} \in \mathcal{C}_Y$ and $\pi^{-1}\mathcal{V} \succeq (\pi^{-1}\mathcal{U})_0^{n-1}$. It follows from $\pi \circ S = T \circ \pi$ that

$$\begin{aligned} \sum_{V \in \mathcal{V}} \sup_{x \in V} e^{n\mathcal{E}(\Delta_x^n)} &= \sum_{V \in \mathcal{V}} \sup_{y \in \pi^{-1}(V)} e^{n\mathcal{E}(\Delta_{\pi y}^n)} = \sum_{V \in \mathcal{V}} \sup_{y \in \pi^{-1}(V)} e^{n\mathcal{E} \circ \pi_*(\Delta_y^n)} \\ &\geq p_n(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}). \end{aligned}$$

Since \mathcal{V} is arbitrary,

$$p_n(T, \mathcal{E}; \mathcal{U}) \geq p_n(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}).$$

On the other hand, by Lemma 2.1,

$$p_n(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}) = \min_{\beta \in \mathcal{P}^*((\pi^{-1}\mathcal{U})_0^{n-1})} \left\{ \sum_{B \in \beta} \sup_{y \in B} e^{n\mathcal{E} \circ \pi_*(\Delta_y^n)} \right\}.$$

Note that if $\beta = \{B_1, B_2, \dots, B_m\} \in \mathcal{P}^*((\pi^{-1}\mathcal{U})_0^{n-1})$, then

$$\pi\beta = \{\pi(B_1), \pi(B_2), \dots, \pi(B_m)\} \in \mathcal{C}_X \text{ and } \pi\beta \succeq \mathcal{U}_0^{n-1}.$$

Hence

$$\sum_{i=1}^m \sup_{y \in B_i} e^{n\mathcal{E} \circ \pi_*(\Delta_y^n)} = \sum_{i=1}^m \sup_{x \in \pi(B_i)} e^{n\mathcal{E}(\Delta_x^n)} \geq p_n(T, \mathcal{E}; \mathcal{U}).$$

By the arbitrariness of β , we have $p_n(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}) \geq p_n(T, \mathcal{E}; \mathcal{U})$.

Therefore, $p_n(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}) = p_n(T, \mathcal{E}; \mathcal{U})$, and the conclusion of the lemma follows. \square

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^o$. We prove Theorem 1.1 by showing

$$\begin{aligned} \sup_{\mu \in \mathcal{M}^e(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\} &\stackrel{\textcircled{1}}{\leq} \underline{P}(T, \mathcal{E}; \mathcal{U}) \stackrel{\textcircled{2}}{\leq} P(T, \mathcal{E}; \mathcal{U}) \\ &\stackrel{\textcircled{3}}{\leq} \sup_{\mu \in \mathcal{M}(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\} \\ &\stackrel{\textcircled{4}}{\leq} \sup_{\mu \in \mathcal{M}^e(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\} \end{aligned} \quad (3.1)$$

Inequality $\textcircled{1}$ is proved in Proposition 3.1. Inequality $\textcircled{2}$ follows from the definitions of $\underline{P}(T, \mathcal{E}; \mathcal{U})$ and $P(T, \mathcal{E}; \mathcal{U})$. Inequality $\textcircled{4}$ is proved in Proposition 3.2 under the assumption that $(T, \mathcal{E}; \mathcal{U})$ has an abundance of ergodic measures. Finally, Inequality $\textcircled{3}$ is proved in Subsection 3.2, Proposition 3.3.

3.1 Proof of inequality $\textcircled{1}$ and $\textcircled{4}$ in (3.1)

We begin by recalling the following basic lemma.

Lemma 3.1 ([29], Lemma 9.9). *Let a_1, a_2, \dots, a_k be given real numbers. If $b_i \geq 0$ for every $1 \leq i \leq k$, and $\sum_{i=1}^k b_i = 1$, then*

$$\sum_{i=1}^k b_i (a_i - \log b_i) \leq \log \left(\sum_{i=1}^k e^{a_i} \right),$$

and equality holds iff $b_i = \frac{e^{a_i}}{\sum_{i=1}^k e^{a_i}}$ for all $i = 1, 2, \dots, k$.

Proposition 3.1. (Inequality $\textcircled{1}$). Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^o$. Then

$$\sup_{\mu \in \mathcal{M}^e(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\} \leq \underline{P}(T, \mathcal{E}; \mathcal{U}).$$

Proof. Fix any $\mu \in \mathcal{M}^e(X, T)$. Since \mathcal{E} is uniformly continuous on $\mathcal{M}(X)$, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $\nu \in \mathcal{M}(X)$ with $W(\nu, \mu) < \delta$, one has $|\mathcal{E}(\nu) - \mathcal{E}(\mu)| < \epsilon$. Since $\mu \in \mathcal{M}^e(X, T)$, for μ -almost every $x \in X$, $\Delta_x^n \rightarrow \mu$ as $n \rightarrow \infty$. Then by Egorov's Theorem, there exists $N \geq 1$ and a measurable subset Γ with $\mu(\Gamma) > 1 - \epsilon$ such that $W(\Delta_x^n, \mu) < \delta$ for every $x \in \Gamma$ and $n \geq N$. It follows that

$$|\mathcal{E}(\Delta_x^n) - \mathcal{E}(\mu)| < \epsilon, \quad x \in \Gamma, n \geq N. \quad (3.2)$$

By Lemma 2.1, for any $n \in \mathbb{N}$, there exists a finite partition $\beta \in \mathcal{P}^*(\mathcal{U}_0^{n-1})$ such that

$$\sum_{B \in \beta} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} = p_n(T, \mathcal{E}; \mathcal{U}).$$

It follows from Lemma 3.1 that

$$\begin{aligned}
\log p_n(T, \mathcal{E}; \mathcal{U}) &= \log \sum_{B \in \beta} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} \\
&\geq \sum_{B \in \beta} \mu(B) \left(\sup_{x \in B} n\mathcal{E}(\Delta_x^n) - \log \mu(B) \right) \\
&= H_\mu(\beta) + \sum_{B \in \beta} \sup_{x \in B} n\mathcal{E}(\Delta_x^n) \cdot \mu(B).
\end{aligned} \tag{3.3}$$

We focus on the latter item. Let $\beta_1 = \{B \in \beta : B \cap \Gamma \neq \emptyset\}$. Then

$$\sum_{B \in \beta_1} \mu(B) \geq \sum_{B \in \beta_1} \mu(B \cap \Gamma) = \mu \left(\bigcup_{B \in \beta_1} (B \cap \Gamma) \right) = \mu(\Gamma) > 1 - \epsilon.$$

Therefore, (3.2) and (3.3) give

$$\begin{aligned}
\log(p_n(T, \mathcal{E}; \mathcal{U})) &\geq H_\mu(\beta) + \sum_{B \in \beta_1} \sup_{x \in B} n\mathcal{E}(\Delta_x^n) \cdot \mu(B) \\
&\geq H_\mu(\beta) + \sum_{B \in \beta_1} n(\mathcal{E}(\mu) - \epsilon) \cdot \mu(B) \\
&\geq H_\mu(\beta) + n(\mathcal{E}(\mu) - \epsilon)(1 - \epsilon) \\
&\geq H_\mu(\mathcal{U}_0^{n-1}) + n(\mathcal{E}(\mu) - \epsilon)(1 - \epsilon),
\end{aligned}$$

where the last inequality follows from $\beta \succeq \mathcal{U}_0^{n-1}$. Dividing the above equation by n and taking the liminf as $n \rightarrow \infty$, one has

$$\underline{P}(T, \mathcal{E}; \mathcal{U}) \geq h_\mu(T, \mathcal{U}) + (\mathcal{E}(\mu) - \epsilon)(1 - \epsilon).$$

By the arbitrariness of ϵ , the result follows. \square

Proposition 3.2. (Inequality ④) Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^0$. Suppose that $(T, \mathcal{E}; \mathcal{U})$ has an abundance of ergodic measures. Then

$$\sup_{\mu \in \mathcal{M}(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\} \leq \sup_{\mu \in \mathcal{M}^e(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\}.$$

Proof. By Definition 1.2, for any $\mu \in \mathcal{M}(X, T)$ and any $\epsilon > 0$, there is an ergodic measure $\nu \in \mathcal{M}^e(X, T)$ such that $h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu) \leq h_\nu(T, \mathcal{U}) + \mathcal{E}(\nu) + \epsilon$, which implies

$$h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu) \leq \sup_{\nu \in \mathcal{M}^e(X, T)} \{h_\nu(T, \mathcal{U}) + \mathcal{E}(\nu)\} + \epsilon.$$

Since $\mu \in \mathcal{M}(X, T)$ and $\epsilon > 0$ are arbitrary,

$$\sup_{\mu \in \mathcal{M}(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\} \leq \sup_{\mu \in \mathcal{M}^e(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\}.$$

\square

3.2 Proof of inequality ③ in (3.1)

To establish Inequality ③, the following lemma is crucial.

Lemma 3.2. *Suppose that $\{\alpha_l\}_{l \geq 1}$ is a sequence of finite partitions of X which are finer than \mathcal{U} . Then for any $\gamma > 0$, there exist $N = N(\gamma) \in \mathbb{N}, E \in \mathbb{R}$ and a sequence of finite subsets $\{D_{n_k}\}_{k \geq 1}$ of X where $n_k \rightarrow \infty$, such that for any $k \geq 1$,*

1. *for all $1 \leq l \leq n_k$, each atom of $(\alpha_l)_0^{n_k-1}$ contains at most one point of D_{n_k} ;*
2. $\sum_{x \in D_{n_k}} e^{n_k \mathcal{E}(\Delta_x^{n_k})} \geq \frac{1}{2n_k N} \cdot e^{n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma)}$;
3. *For any $\mu \in \mathcal{M}(X)$ that is a convex combination of $\{\Delta_x^{n_k}\}_{x \in D_{n_k}}$, we have*

$$|\mathcal{E}(\mu) - E| \leq \gamma.$$

Proof. We begin by proving the following claim.

Claim. For any $n \geq 1$, there exists a finite subset B_n of X such that for any $1 \leq l \leq n$, each atom of $(\alpha_l)_0^{n-1}$ contains at most one point of B_n . Moreover,

$$\sum_{x \in B_n} e^{n \mathcal{E}(\Delta_x^n)} \geq \frac{1}{2n} \cdot p_n(T, \mathcal{E}; \mathcal{U}).$$

Proof of the Claim. Fix any $n \in \mathbb{N}$. For each $x \in X$, denote by $A_l(x)$ the atom of $(\alpha_l)_0^{n-1}$ which contains x . We now construct the set B_n . Choose $x_1 \in X$ such that $e^{n \mathcal{E}(\Delta_{x_1}^n)} = \max_{x \in X} e^{n \mathcal{E}(\Delta_x^n)}$ and denote $X_1 = X$. By induction, suppose that X_i and $x_i \in X_i$ have been defined. Let $X_{i+1} = X_i \setminus \bigcup_{l=1}^n A_l(x_i)$, and choose $x_{i+1} \in X_{i+1}$ such that $e^{n \mathcal{E}(\Delta_{x_{i+1}}^n)} \geq \frac{1}{2} \sup_{x \in X_{i+1}} e^{n \mathcal{E}(\Delta_x^n)}$. Since $\{A_l(x) : 1 \leq l \leq n, x \in X\}$ is a finite set, there exists $m \geq 1$ such that $X_m \setminus \bigcup_{l=1}^n A_l(x_m) = \emptyset$. Then we obtain a finite set $B_n := \{x_1, \dots, x_m\}$ such that for each $1 \leq i \leq m$,

- (1) $e^{n \mathcal{E}(\Delta_{x_i}^n)} \geq \frac{1}{2} \sup_{x \in X_i} e^{n \mathcal{E}(\Delta_x^n)}$;
- (2) $X_i = X_{i+1} \cup (\bigcup_{l=1}^n A_l(x_i) \cap X_i)$, where $X_{m+1} = \emptyset$.

By construction, for all $1 \leq l \leq n$, each atom of $(\alpha_l)_0^{n-1}$ contains at most one point of B_n . Moreover,

$$e^{n \mathcal{E}(\Delta_{x_i}^n)} \geq \frac{1}{2} \sup_{x \in X_i} e^{n \mathcal{E}(\Delta_x^n)} \geq \frac{1}{2} \sup_{x \in A_l(x_i) \cap X_i} e^{n \mathcal{E}(\Delta_x^n)}, \quad \forall 1 \leq l \leq n.$$

Let $\beta = \{A_l(x_i) \cap X_i : 1 \leq l \leq n, 1 \leq i \leq m\}$. Then $\beta \in \mathcal{C}_X$ and $\beta \succeq \mathcal{U}_0^{n-1}$. Therefore,

$$\begin{aligned} \sum_{x \in B_n} e^{n\mathcal{E}(\Delta_x^n)} &= \sum_{i=1}^m e^{n\mathcal{E}(\Delta_{x_i}^n)} \geq \sum_{i=1}^m \frac{1}{n} \sum_{l=1}^n \frac{1}{2} \sup_{x \in A_l(x_i) \cap X_i} e^{n\mathcal{E}(\Delta_x^n)} \\ &= \frac{1}{2n} \sum_{B \in \beta} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} \\ &\geq \frac{1}{2n} \cdot p_n(T, \mathcal{E}; \mathcal{U}). \end{aligned}$$

□

We now continue the proof of Lemma 3.2. Since \mathcal{E} is uniformly continuous on $\mathcal{M}(X)$, for any $\gamma > 0$, there exists $\delta > 0$ such that for all $\mu, \nu \in \mathcal{M}(X)$ satisfying $W(\mu, \nu) \leq \delta$, one has $|\mathcal{E}(\mu) - \mathcal{E}(\nu)| \leq \gamma$.

Let $S = \{v_1, v_2, \dots, v_N\}$ be a δ -dense subset of $\mathcal{M}(X)$. For each $1 \leq i \leq N$, denote $B(v_i, \delta) = \{\mu \in \mathcal{M}(X) : W(\mu, v_i) \leq \delta\}$. Set $V_1 = B(v_1, \delta)$ and $V_i = B(v_i, \delta) \setminus \bigcup_{j=1}^{i-1} V_j$ for $2 \leq i \leq N$. Then $\{V_i\}_{1 \leq i \leq N}$ is a partition of $\mathcal{M}(X)$, and $|\mathcal{E}(\mu) - \mathcal{E}(v_i)| \leq \gamma$ for all $\mu \in V_i$.

Let $\{n_k\}_{k \geq 1}$ be a sequence of integers diverging to infinity such that

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \log p_{n_k}(T, \mathcal{E}; \mathcal{U}) = P(T, \mathcal{E}; \mathcal{U}).$$

Then there exists $K_0 \in \mathbb{N}$ such that $p_{n_k}(T, \mathcal{E}; \mathcal{U}) \geq e^{n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma)}$ holds for every $k \geq K_0$. Denote $C_{n_k, i} = \{x \in B_{n_k} : \Delta_x^{n_k} \in V_i\}$. Then by the Claim above, for every $k \geq K_0$,

$$\begin{aligned} \sum_{x \in B_{n_k}} e^{n_k \mathcal{E}(\Delta_x^{n_k})} &= \sum_{i=1}^N \sum_{x \in C_{n_k, i}} e^{n_k \mathcal{E}(\Delta_x^{n_k})} \geq \frac{1}{2n_k} \cdot p_{n_k}(T, \mathcal{E}; \mathcal{U}) \\ &\geq \frac{1}{2n_k} \cdot e^{n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma)}. \end{aligned}$$

It follows that for each $k \geq K_0$ there exists $1 \leq i \leq N$ such that

$$\sum_{x \in C_{n_k, i}} e^{n_k \mathcal{E}(\Delta_x^{n_k})} \geq \frac{1}{2n_k N} \cdot e^{n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma)}.$$

Therefore, there exists $i_0 \in [1, N]$ and a subsequence of $\{n_k\}_{k \geq 1}$ (still denoted by $\{n_k\}_{k \geq 1}$) such that

$$\sum_{x \in C_{n_k, i_0}} e^{n_k \mathcal{E}(\Delta_x^{n_k})} \geq \frac{1}{2n_k N} \cdot e^{n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma)}, \quad \forall k \geq 1.$$

Denote $D_{n_k} = C_{n_k, i_0}$ and $E = \mathcal{E}(v_{i_0})$. Then D_{n_k} satisfies (i) and (ii) of Lemma 3.2. To check (iii), suppose that a probability measure $\mu = \sum_{x \in D_{n_k}} \lambda_x \Delta_x^{n_k}$ is a convex combination of the measures $\Delta_x^{n_k}$, then it follows from $W(\Delta_x^{n_k}, v_{i_0}) \leq \delta$ for every $x \in D_{n_k}$ that $W(\mu, v_{i_0}) \leq \delta$. Hence $|\mathcal{E}(\mu) - E| \leq |\mathcal{E}(\mu) - \mathcal{E}(v_{i_0})| \leq \gamma$. □

Now we are ready to prove inequality ③ in (3.1).

Proposition 3.3. (Inequality ③). Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^0$. Then

$$P(T, \mathcal{E}; \mathcal{U}) \leq \sup_{\mu \in \mathcal{M}(X, T)} \{h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu)\}.$$

Proof. We split the proof into two steps.

Step 1. Suppose that (X, T) is an invertible, zero-dimensional TDS (i.e., it has a topological base consisting of clopen subsets). Denote $\mathcal{U} = \{U_1, U_2, \dots, U_d\}$ and let

$$\mathcal{U}^* = \{\alpha \in \mathcal{P}_X : \alpha = \{A_1, A_2, \dots, A_d\}, A_i \subset U_i, i = 1, 2, \dots, d\}.$$

Since X is zero-dimensional, \mathcal{U}^* forms a countable set of partitions. Moreover, these partitions are finer than \mathcal{U} and consist entirely of clopen sets. Denote $\mathcal{U}^* = \{\alpha_l : l \geq 1\}$.

Then by Lemma 3.2, for any $\gamma > 0$, there exist $N = N(\gamma) \in \mathbb{N}, E \in \mathbb{R}$ and a sequence of finite subsets $\{D_{n_k}\}_{k \geq 1}$ of X where $n_k \rightarrow \infty$, such that for any $k \geq 1$,

1. for all $1 \leq l \leq n_k$, each atom of $(\alpha_l)_{n_k}^{n_k-1}$ contains at most one point of D_{n_k} ;
2. $\sum_{x \in D_{n_k}} e^{n_k \mathcal{E}(\Delta_x^{n_k})} \geq \frac{1}{2n_k N} \cdot e^{n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma)}$;
3. if $\mu \in \mathcal{M}(X)$ is a convex combination of the measures $\Delta_x^{n_k}$ where x runs over D_{n_k} , then $|\mathcal{E}(\mu) - E| \leq \gamma$.

Let

$$\nu_k := \sum_{x \in D_{n_k}} \lambda_{n_k}(x) \delta_x,$$

where $\lambda_{n_k}(x) = \frac{e^{n_k \mathcal{E}(\Delta_x^{n_k})}}{\sum_{y \in D_{n_k}} e^{n_k \mathcal{E}(\Delta_y^{n_k})}}$. Define the empirical measure

$$\mu_k := \frac{1}{n_k} \sum_{i=0}^{n_k-1} T_*^i \nu_k = \sum_{x \in D_{n_k}} \lambda_{n_k}(x) \Delta_x^{n_k}.$$

Up to extracting a further subsequence, we may assume $\mu_\infty = \lim_{k \rightarrow \infty} \mu_k$. Then $\mu_\infty \in \mathcal{M}(X, T)$.

Moreover, by $|\mathcal{E}(\mu_k) - E| \leq \gamma$ for every $k \geq 1$, we also obtain

$$|\mathcal{E}(\mu_\infty) - E| \leq \gamma. \quad (3.4)$$

Since T is invertible, by Proposition 2.1,

$$h_{\mu_\infty}(T, \mathcal{U}) = h_{\mu_\infty}^+(T, \mathcal{U}) = \inf_{\alpha \in \mathcal{U}^*} h_{\mu_\infty}(T, \alpha) = \inf_{l \geq 1} h_{\mu_\infty}(T, \alpha_l).$$

It suffices to prove for each $l \geq 1$:

$$P(T, \mathcal{E}; \mathcal{U}) \leq h_{\mu_\infty}(T, \alpha_l) + \mathcal{E}(\mu_\infty) + 3\gamma.$$

Since each atom of $(\alpha_l)_0^{n_k-1}$ contains at most one point of D_{n_k} , it follows that

$$H_{v_k} \left((\alpha_l)_0^{n_k-1} \right) = \sum_{x \in D_{n_k}} \lambda_{n_k}(x) \log \frac{1}{\lambda_{n_k}(x)}.$$

By Lemma 3.2 and inequality (3.4), we have

$$\begin{aligned} \log \frac{1}{\lambda_{n_k}(x)} &\geq \log \frac{1}{2Nn_k} + n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma) - n_k \mathcal{E}(\Delta_x^{n_k}) \\ &\geq n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma) - n_k \cdot (E + \gamma) - \log(2Nn_k) \\ &\geq n_k(P(T, \mathcal{E}; \mathcal{U}) - \gamma) - n_k \cdot (\mathcal{E}(\mu_\infty) + 2\gamma) - \log(2Nn_k). \end{aligned}$$

Therefore,

$$\frac{1}{n_k} H_{v_k} \left((\alpha_l)_0^{n_k-1} \right) \geq P(T, \mathcal{E}; \mathcal{U}) - \mathcal{E}(\mu_\infty) - 3\gamma - \frac{\log(2Nn_k)}{n_k}. \quad (3.5)$$

Fix $m, n_k \in \mathbb{N}$ with $n_k > l$ and $1 \leq m \leq n_k - 1$. Let $a(j) = \lfloor \frac{n_k - j}{m} \rfloor, j = 0, 1, \dots, m-1$. Then

$$\bigvee_{i=0}^{n_k-1} T^{-i} \alpha_l = \bigvee_{r=0}^{a(j)-1} T^{-(mr+j)} (\alpha_l)_0^{m-1} \vee \bigvee_{t \in S_j} T^{-t} \alpha_l,$$

where $S_j = \{0, 1, \dots, j-1\} \cup \{j+ma(j), \dots, n_k-1\}$. Since $|S_j| \leq 2m$, by the sub-additivity of $H_{v_k}(\cdot)$,

$$\begin{aligned} H_{v_k} \left((\alpha_l)_0^{n_k-1} \right) &\leq \sum_{r=0}^{a(j)-1} H_{v_k} \left(T^{-(mr+j)} (\alpha_l)_0^{m-1} \right) + H_{v_k} \left(\bigvee_{l \in S_j} T^{-l} \alpha_l \right) \\ &\leq \sum_{r=0}^{a(j)-1} H_{T_*^{(mr+j)} v_k} \left((\alpha_l)_0^{m-1} \right) + 2m \log d, \end{aligned}$$

where $d = |\mathcal{U}|$ represents the cardinality of \mathcal{U} . Summing j from 0 to $m-1$, we obtain

$$\begin{aligned} H_{v_k} \left((\alpha_l)_0^{n_k-1} \right) &\leq \frac{1}{m} \sum_{j=0}^{m-1} \sum_{r=0}^{a(j)-1} H_{T_*^{(mr+j)} v_k} \left((\alpha_l)_0^{m-1} \right) + 2m \log d \\ &\leq \frac{1}{m} \sum_{j=0}^{n_k-1} H_{T_*^j v_k} \left((\alpha_l)_0^{m-1} \right) + 2m \log d. \end{aligned}$$

Due to the concavity of $H_{\{\cdot\}}((\alpha_l)_0^{m-1})$ over $\mathcal{M}(X)$,

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} H_{T_*^j v_k}((\alpha_l)_0^{m-1}) \leq H_{\mu_k}((\alpha_l)_0^{m-1}).$$

We conclude that

$$\frac{1}{n_k} H_{v_k}((\alpha_l)_0^{n_k-1}) \leq \frac{1}{m} H_{\mu_k}((\alpha_l)_0^{m-1}) + \frac{2m \log d}{n_k}. \quad (3.6)$$

From (3.5) and (3.6), we obtain

$$P(T, \mathcal{E}; \mathcal{U}) - \mathcal{E}(\mu_\infty) - 3\gamma - \frac{\log(2Nn_k)}{n_k} \leq \frac{1}{m} H_{\mu_k}((\alpha_l)_0^{m-1}) + \frac{2m \log d}{n_k}.$$

Since α_l is clopen, taking $k \rightarrow \infty$, one has

$$P(T, \mathcal{E}; \mathcal{U}) - \mathcal{E}(\mu_\infty) - 3\gamma \leq \frac{1}{m} H_{\mu_\infty}((\alpha_l)_0^{m-1}).$$

The result follows by letting $m \rightarrow \infty$.

Step 2. In the general case where (X, T) is not zero-dimensional. Recall that (X, T) always has a zero-dimensional extension (see [30]). Let (Y, S) be an invertible zero-dimensional TDS and $\pi: Y \rightarrow X$ be the factor map from (Y, S) to (X, T) .

Since $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ is an energy and $\mathcal{U} \in \mathcal{C}_X^0$, one has $\mathcal{E} \circ \pi_*: \mathcal{M}(Y) \rightarrow \mathbb{R}$ is an energy and $\pi^{-1}\mathcal{U} \in \mathcal{C}_Y^0$. Then by the previous case,

$$P(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}) \leq \sup_{v \in \mathcal{M}_S(Y)} \left\{ h_v(S, \pi^{-1}\mathcal{U}) + \mathcal{E} \circ \pi_*(v) \right\}.$$

Note that $\pi_*(\mathcal{M}_S(Y)) \subset \mathcal{M}(X, T)$. By Lemma 2.2 and 2.3,

$$\begin{aligned} P(T, \mathcal{E}; \mathcal{U}) &= P(S, \mathcal{E} \circ \pi_*; \pi^{-1}\mathcal{U}) \\ &\leq \sup_{v \in \mathcal{M}_S(Y)} \left\{ h_v(S, \pi^{-1}\mathcal{U}) + \mathcal{E} \circ \pi_*(v) \right\} \\ &= \sup_{v \in \mathcal{M}_S(Y)} \left\{ h_{\pi_* v}(T, \mathcal{U}) + \mathcal{E}(\pi_* v) \right\} \\ &\leq \sup_{\mu \in \mathcal{M}(X, T)} \left\{ h_\mu(T, \mathcal{U}) + \mathcal{E}(\mu) \right\}. \end{aligned}$$

The proof is complete. □

4 Relationship between nonlinear topological pressure and nonlinear local pressure

In this section, we give several equivalent definitions of nonlinear topological pressure.

Let (X, T) be a TDS. For $n \in \mathbb{N}, \varepsilon > 0$ and $x \in X$, denote

$$B_n(x, \varepsilon) = \{y \in X : d(T^i x, T^i y) < \varepsilon, \forall 0 \leq i \leq n-1\}.$$

A set $E \subset X$ is said to be an (n, ε) -separated set of X with respect to T , if for any distinct $x, y \in E$, $y \notin B_n(x, \varepsilon)$. A set $D \subset X$ is said to be an (n, ε) -spanning set of X with respect to T , if for any $x \in X$, there exists $y \in D$ such that $y \in B_n(x, \varepsilon)$.

Definition 4.1. Let $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy. For $n \geq 1$ and $\varepsilon > 0$, let

$$P_n(T, \mathcal{E}, \varepsilon) = \sup \left\{ \sum_{x \in E} e^{n\mathcal{E}(\Delta_x^n)} : E \text{ is an } (n, \varepsilon)\text{-separated set of } X \right\}.$$

Then the upper and lower nonlinear topological pressures are defined respectively by

$$P(T, \mathcal{E}) = \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(T, \mathcal{E}, \varepsilon),$$

$$\underline{P}(T, \mathcal{E}) = \liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_n(T, \mathcal{E}, \varepsilon).$$

Definition 4.2. Let $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy. For $n \geq 1$ and $\varepsilon > 0$, let

$$Q_n(T, \mathcal{E}, \varepsilon) = \inf \left\{ \sum_{x \in F} e^{n\mathcal{E}(\Delta_x^n)} : F \text{ is an } (n, \varepsilon)\text{-spanning set of } X \right\}.$$

Then we define

$$Q(T, \mathcal{E}) = \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log Q_n(T, \mathcal{E}, \varepsilon),$$

$$\underline{Q}(T, \mathcal{E}) = \liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log Q_n(T, \mathcal{E}, \varepsilon).$$

We also give several distinct definitions of nonlinear local pressure.

Definition 4.3. Let $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^0$. For $n \geq 1$, denote

$$p_n^1(T, \mathcal{E}; \mathcal{U}) = \inf \left\{ \sum_{V \in \mathcal{V}} \sup_{x \in V} e^{n\mathcal{E}(\Delta_x^n)} : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}_0^{n-1} \right\},$$

$$p_n^2(T, \mathcal{E}; \mathcal{U}) = \inf \left\{ \sum_{V \in \mathcal{V}} \inf_{x \in V} e^{n\mathcal{E}(\Delta_x^n)} : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}_0^{n-1} \right\}.$$

Then we define

$$P_1(T, \mathcal{E}; \mathcal{U}) = P(T, \mathcal{E}; \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log p_n^1(T, \mathcal{E}; \mathcal{U}),$$

$$\underline{P}_1(T, \mathcal{E}; \mathcal{U}) = \underline{P}(T, \mathcal{E}; \mathcal{U}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log p_n^1(T, \mathcal{E}; \mathcal{U}),$$

and

$$P_2(T, \mathcal{E}; \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log p_n^2(T, \mathcal{E}; \mathcal{U}),$$

$$\underline{P}_2(T, \mathcal{E}; \mathcal{U}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log p_n^2(T, \mathcal{E}; \mathcal{U}).$$

Definition 4.4. Let $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^0$. For $n \geq 1$, denote

$$p_n^3(T, \mathcal{E}; \mathcal{U}) = \inf \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} : \beta \text{ is a finite subcover of } \mathcal{U}_0^{n-1} \right\},$$

$$p_n^4(T, \mathcal{E}; \mathcal{U}) = \inf \left\{ \sum_{B \in \beta} \inf_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} : \beta \text{ is a finite subcover of } \mathcal{U}_0^{n-1} \right\}.$$

Then we define

$$P_3(T, \mathcal{E}; \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log p_n^3(T, \mathcal{E}; \mathcal{U}),$$

$$\underline{P}_3(T, \mathcal{E}; \mathcal{U}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log p_n^3(T, \mathcal{E}; \mathcal{U}),$$

and

$$P_4(T, \mathcal{E}; \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log p_n^4(T, \mathcal{E}; \mathcal{U}),$$

$$\underline{P}_4(T, \mathcal{E}; \mathcal{U}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log p_n^4(T, \mathcal{E}; \mathcal{U}).$$

Lemma 4.1. Let (X, T) be a TDS, $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy and $\mathcal{U} \in \mathcal{C}_X^0$. Then for any $n \geq 1$,

$$p_n^2(T, \mathcal{E}; \mathcal{U}) \leq p_n^4(T, \mathcal{E}; \mathcal{U}) \leq p_n^3(T, \mathcal{E}; \mathcal{U}),$$

$$p_n^2(T, \mathcal{E}; \mathcal{U}) \leq p_n^1(T, \mathcal{E}; \mathcal{U}) \leq p_n^3(T, \mathcal{E}; \mathcal{U}).$$

Proof. Note that if β is a finite subcover of \mathcal{U}_0^{n-1} , then $\beta \in \mathcal{C}_X$ and $\beta \succeq \mathcal{U}_0^{n-1}$. This immediately implies $p_n^2(T, \mathcal{E}; \mathcal{U}) \leq p_n^4(T, \mathcal{E}; \mathcal{U})$ and $p_n^1(T, \mathcal{E}; \mathcal{U}) \leq p_n^3(T, \mathcal{E}; \mathcal{U})$. The remaining inequalities $p_n^4(T, \mathcal{E}; \mathcal{U}) \leq p_n^3(T, \mathcal{E}; \mathcal{U})$ and $p_n^2(T, \mathcal{E}; \mathcal{U}) \leq p_n^1(T, \mathcal{E}; \mathcal{U})$ follow directly from comparing the infimum/supremum operations in their definitions. \square

Lemma 4.2. Let (X, T) be a TDS, $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy. For any $\varepsilon > 0$, let $\mathcal{U} \in \mathcal{C}_X^0$ with $\text{diam}(\mathcal{U}) \leq \varepsilon$ and denote

$$\tau_\varepsilon = \sup \{ |\mathcal{E}(\mu) - \mathcal{E}(\nu)| : \mu, \nu \in \mathcal{M}(X), W(\mu, \nu) \leq \varepsilon \}.$$

1. If $\delta > 0$ is the Lebesgue number of \mathcal{U} , then for any $n \geq 1$,

$$e^{-n\tau_\varepsilon} \cdot p_n^3(T, \mathcal{E}; \mathcal{U}) \leq Q_n(T, \mathcal{E}, \delta/2);$$

2. For any $n \geq 1$, one has

$$Q_n(T, \mathcal{E}, \varepsilon) \leq P_n(T, \mathcal{E}, \varepsilon) \leq e^{n\tau_\varepsilon} \cdot p_n^2(T, \mathcal{E}; \mathcal{U}).$$

Proof. (i) Let $\mathcal{U} \in \mathcal{C}_X^0$ with $\text{diam}(\mathcal{U}) \leq \varepsilon$. For any $B \in \mathcal{U}_0^{n-1}$ and any $x, y \in B$, we estimate the Wasserstein distance between empirical measures:

$$\begin{aligned} W(\Delta_x^n, \Delta_y^n) &= \sup \left\{ \Delta_x^n(g) - \Delta_y^n(g) : g : X \rightarrow \mathbb{R} \text{ is 1-Lipschitz} \right\} \\ &= \sup \left\{ \frac{\sum_{i=0}^{n-1} g(T^i x) - g(T^i y)}{n} : g : X \rightarrow \mathbb{R} \text{ is 1-Lipschitz} \right\} \\ &\leq \sum_{i=0}^{n-1} \frac{d(T^i x, T^i y)}{n} \\ &\leq \varepsilon. \end{aligned}$$

Therefore, by the definition of $p_n^1(T, \mathcal{E}; \mathcal{U})$ and τ_ε ,

$$p_n^1(T, \mathcal{E}; \mathcal{U}) \leq e^{n\tau_\varepsilon} \cdot p_n^2(T, \mathcal{E}; \mathcal{U}), \quad (4.1)$$

and

$$p_n^3(T, \mathcal{E}; \mathcal{U}) \leq e^{n\tau_\varepsilon} \cdot p_n^4(T, \mathcal{E}; \mathcal{U}). \quad (4.2)$$

Now let F be an $(n, \delta/2)$ -spanning set of X . Then

$$X = \bigcup_{x \in F} B_n(x, \delta/2).$$

Since δ is the Lebesgue number of \mathcal{U} , each $B(T^i x, \delta/2)$ is contained in some element of \mathcal{U} . Thus $B_n(x, \delta/2) = \bigcap_{i=0}^{n-1} T^{-i} B(T^i x, \delta/2)$ is contained in some element of \mathcal{U}_0^{n-1} . It follows that

$$p_n^4(T, \mathcal{E}; \mathcal{U}) \leq \sum_{x \in F} e^{n\mathcal{E}(\Delta_x^n)}. \quad (4.3)$$

Combining with (4.2), we obtain

$$e^{-n\tau_\varepsilon} \cdot p_n^3(T, \mathcal{E}; \mathcal{U}) \leq Q_n(T, \mathcal{E}, \delta/2).$$

(ii) Let E be an (n, ε) -separated set of X with maximum cardinality. Then E is also (n, ε) -spanning, giving

$$Q_n(T, \mathcal{E}, \varepsilon) \leq P_n(T, \mathcal{E}, \varepsilon).$$

For $\mathcal{U} \in \mathcal{C}_X^0$ with $\text{diam}(\mathcal{U}) \leq \varepsilon$, each element of \mathcal{U}_0^{n-1} contains at most one point in E . Hence

$$\sum_{x \in E} e^{n\mathcal{E}(\Delta_x^n)} \leq p_n^1(T, \mathcal{E}; \mathcal{U}).$$

Applying (4.1) completes the proof:

$$P_n(T, \mathcal{E}, \varepsilon) \leq e^{n\tau_\varepsilon} \cdot p_n^2(T, \mathcal{E}; \mathcal{U}).$$

□

Theorem 4.1. *Let (X, T) be a TDS, $\mathcal{E} : \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy. Then the following quantities all coincide with $P(T, \mathcal{E})$.*

1. $Q(T, \mathcal{E})$;
2. $\limsup_{\varepsilon \rightarrow 0} \sup_{\mathcal{U} \in \mathcal{C}_X^0} \{P_i(T, \mathcal{E}; \mathcal{U}) : \text{diam}(\mathcal{U}) \leq \varepsilon\}$, $\forall i \in \{1, 2, 3, 4\}$;
3. $\lim_{k \rightarrow \infty} P_i(T, \mathcal{E}; \mathcal{U}_k)$, $\forall \{\mathcal{U}_k\} \subset \mathcal{C}_X^0$ with $\text{diam}(\mathcal{U}_k) \rightarrow 0$, $\forall i \in \{1, 2, 3, 4\}$;
4. $\sup_{\mathcal{U} \in \mathcal{C}_X^0} P_i(T, \mathcal{E}; \mathcal{U})$, $\forall i \in \{1, 2, 4\}$.

Proof. (i) & (ii) Let $\varepsilon > 0$ and $\mathcal{U} \in \mathcal{C}_X^0$ with $\text{diam}(\mathcal{U}) \leq \varepsilon$. Let $\delta > 0$ be the Lebesgue number of \mathcal{U} . Then by Lemma 4.2(i),

$$e^{-n\tau_\varepsilon} \cdot p_n^3(T, \mathcal{E}; \mathcal{U}) \leq Q_n(T, \mathcal{E}, \delta/2), \quad \forall n \geq 1.$$

Consequently, for any $\mathcal{U} \in \mathcal{C}_X^0$ with $\text{diam}(\mathcal{U}) \leq \varepsilon$,

$$-\tau_\varepsilon + P_3(T, \mathcal{E}; \mathcal{U}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log Q_n(T, \mathcal{E}, \delta/2) \leq Q(T, \mathcal{E}).$$

Since $\tau_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\mathcal{U} \in \mathcal{C}_X^0} \{P_3(T, \mathcal{E}; \mathcal{U}) : \text{diam}(\mathcal{U}) \leq \varepsilon\} \leq Q(T, \mathcal{E}). \quad (4.4)$$

Similarly, by Lemma 4.2(ii), we may also obtain

$$Q(T, \mathcal{E}) \leq P(T, \mathcal{E}) \leq \limsup_{\varepsilon \rightarrow 0} \sup_{\mathcal{U} \in \mathcal{C}_X^0} \{P_2(T, \mathcal{E}; \mathcal{U}) : \text{diam}(\mathcal{U}) \leq \varepsilon\}. \quad (4.5)$$

By Lemma 4.1,

$$P_2(T, \mathcal{E}; \mathcal{U}) \leq P_4(T, \mathcal{E}; \mathcal{U}) \leq P_3(T, \mathcal{E}; \mathcal{U}), \quad (4.6)$$

and

$$P_2(T, \mathcal{E}; \mathcal{U}) \leq P_1(T, \mathcal{E}; \mathcal{U}) \leq P_3(T, \mathcal{E}; \mathcal{U}). \quad (4.7)$$

Combining (4.4)-(4.7), we conclude

$$P(T, \mathcal{E}) = Q(T, \mathcal{E}) = \limsup_{\varepsilon \rightarrow 0} \sup_{\mathcal{U} \in \mathcal{C}_X^0} \{P_i(T, \mathcal{E}; \mathcal{U}) : \text{diam}(\mathcal{U}) \leq \varepsilon\}, \quad \forall i \in \{1, 2, 3, 4\}.$$

The same argument proves (iii).

(iv) For any $\mathcal{U} \in \mathcal{C}_X^0$, suppose that δ is the Lebesgue number of \mathcal{U} . Let F be an $(n, \delta/2)$ -spanning set of X . Then by (4.3) and Lemma 4.1,

$$p_n^2(T, \mathcal{E}; \mathcal{U}) \leq p_n^4(T, \mathcal{E}; \mathcal{U}) \leq Q_n(T, \mathcal{E}, \delta/2) \leq P(T, \mathcal{E}).$$

Hence,

$$\sup_{\mathcal{U} \in \mathcal{C}_X^0} P_2(T, \mathcal{E}; \mathcal{U}) \leq \sup_{\mathcal{U} \in \mathcal{C}_X^0} P_4(T, \mathcal{E}; \mathcal{U}) \leq P(T, \mathcal{E}).$$

From part (ii), for $i \in \{1, 2\}$,

$$P(T, \mathcal{E}) = \limsup_{\varepsilon \rightarrow 0} \sup_{\mathcal{U} \in \mathcal{C}_X^0} \{P_i(T, \mathcal{E}; \mathcal{U}) : \text{diam}(\mathcal{U}) \leq \varepsilon\} \leq \sup_{\mathcal{U} \in \mathcal{C}_X^0} P_i(T, \mathcal{E}; \mathcal{U}).$$

Therefore,

$$\sup_{\mathcal{U} \in \mathcal{C}_X^0} P_2(T, \mathcal{E}; \mathcal{U}) = \sup_{\mathcal{U} \in \mathcal{C}_X^0} P_4(T, \mathcal{E}; \mathcal{U}) = P(T, \mathcal{E}) \leq \sup_{\mathcal{U} \in \mathcal{C}_X^0} P_1(T, \mathcal{E}; \mathcal{U}).$$

It's left to prove

$$P(T, \mathcal{E}) \geq \sup_{\mathcal{U} \in \mathcal{C}_X^0} P_1(T, \mathcal{E}; \mathcal{U}).$$

Fix an open cover $\mathcal{U} \in \mathcal{C}_X^0$ with Lebesgue number δ , and let $n \geq 1$ be an arbitrary integer. Choose $x_1 \in X$ such that $e^{n\mathcal{E}(\Delta_{x_1}^n)} = \sup_{x \in X} e^{n\mathcal{E}(\Delta_x^n)}$ and select $U_{i_1} \in \mathcal{U}_0^{n-1}$ satisfying $B_n(x_1, \delta/2) \subset U_{i_1}$. Denote $V_1 = U_{i_1}$. Next, pick $x_2 \in X \setminus V_1$ such that

$$e^{n\mathcal{E}(\Delta_{x_2}^n)} = \sup_{x \in X \setminus V_1} e^{n\mathcal{E}(\Delta_x^n)},$$

and take $U_{i_2} \in \mathcal{U}_0^{n-1}$ with $B_n(x_2, \delta/2) \subset U_{i_2}$. Let $V_2 = U_{i_2} \setminus U_{i_1}$. Since X is compact, we can repeat this procedure inductively to obtain a finite set $E = \{x_1, \dots, x_m\}$, a collection $\{U_{i_k}\}_{k=1}^m \subset \mathcal{U}_0^{n-1}$ and a disjoint cover $\mathcal{V} = \{V_1, \dots, V_m\}$ such that for every $1 \leq k \leq m$,

1. $B_n(x_k, \delta/2) \subset U_{i_k}$, $x_k \in V_k = U_{i_k} \setminus \bigcup_{j=1}^{k-1} U_{i_j}$, and $X = \bigcup_{k=1}^m V_k$;

2. $e^{n\mathcal{E}(\Delta_{x_k}^n)} = \sup_{x \in X \setminus \bigcup_{j=1}^{k-1} V_j} e^{n\mathcal{E}(\Delta_x^n)}$.

By construction, E is an $(n, \delta/2)$ -separated set of X , $\mathcal{V} \in \mathcal{C}_X$ and $\mathcal{V} \succeq \mathcal{U}_0^{n-1}$. Moreover,

$$\sum_{V \in \mathcal{V}} \sup_{x \in V} e^{n\mathcal{E}(\Delta_x^n)} = \sum_{k=1}^m e^{n\mathcal{E}(\Delta_{x_k}^n)} = \sum_{x \in E} e^{n\mathcal{E}(\Delta_x^n)}.$$

It follows that

$$p_n^1(T, \mathcal{E}; \mathcal{U}) \leq P_n(T, \mathcal{E}, \delta/2) \leq P(T, \mathcal{E}).$$

Therefore,

$$P_1(T, \mathcal{E}; \mathcal{U}) \leq P(T, \mathcal{E}), \quad \forall \mathcal{U} \in \mathcal{C}_X^o,$$

which completes the proof. \square

Remark 4.1. There exist cases where $\sup_{\mathcal{U} \in \mathcal{C}_X^o} P_3(T, \mathcal{E}; \mathcal{U}) > P(T, \mathcal{E})$. As an example, consider the space $X = \{p\} \sqcup \Sigma_2$, where $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ is the full two-sided shift space. Let $T: X \rightarrow X$ be defined by $T(p) = p$, and $T|_{\Sigma_2}$ acts as the left shift map. Let $f: X \rightarrow \mathbb{R}$ be a function defined by $f(p) = 10$ and $f|_{\Sigma_2} \equiv 0$. The energy \mathcal{E} is defined by $\mathcal{E}(\mu) = \int f d\mu$. Then

$$\begin{aligned} P(T, \mathcal{E}) &= \sup_{\mu \in \mathcal{M}^e(X, T)} \left\{ h_\mu(T) + \int f d\mu \right\} \\ &= \max \left\{ h_{\mu_p}(T) + \int f d\mu_p, \sup_{\mu \in \mathcal{M}^e(\Sigma_2, T)} \left\{ h_\mu(T) + \int f d\mu \right\} \right\} \\ &= 10. \end{aligned}$$

However, if we choose $\mathcal{U} = \{\{p\} \cup [0], \{p\} \cup [1]\}$, where $[0], [1]$ are cylinder sets in Σ_2 , then

$$p_n^3(T, \mathcal{E}; \mathcal{U}) = \sum_{B \in \mathcal{U}_0^{n-1}} \sup_{x \in B} e^{n\mathcal{E}(\Delta_x^n)} = 2^n \cdot e^{n \cdot f(p)}.$$

As a consequence,

$$P_3(T, \mathcal{E}; \mathcal{U}) = \log 2 + 10 > P(T, \mathcal{E}).$$

By replacing all instances of \limsup with \liminf in the proof of Theorem 4.1, we immediately obtain the following analogous result. As the argument requires no substantive changes beyond this substitution, we state the result without proof.

Theorem 4.2. Let (X, T) be a TDS, $\mathcal{E}: \mathcal{M}(X) \rightarrow \mathbb{R}$ be an energy. Then each of the following equals $\underline{P}(T, \mathcal{E})$.

1. $\underline{Q}(T, \mathcal{E})$;
2. $\limsup_{\delta \rightarrow 0} \sup_{\mathcal{U} \in \mathcal{C}_X^o} \{ \underline{P}_i(T, \mathcal{E}; \mathcal{U}) : \text{diam}(\mathcal{U}) \leq \delta \}, \quad \forall i \in \{1, 2, 3, 4\}$;
3. $\lim_{k \rightarrow \infty} \underline{P}_i(T, \mathcal{E}; \mathcal{U}_k), \quad \forall \{\mathcal{U}_k\} \subset \mathcal{C}_X^o$ with $\text{diam}(\mathcal{U}_k) \rightarrow 0, \forall i \in \{1, 2, 3, 4\}$;
4. $\sup_{\mathcal{U} \in \mathcal{C}_X^o} \underline{P}_i(T, \mathcal{E}; \mathcal{U}), \quad \forall i \in \{1, 2, 4\}$.

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Conflicts of Interest

The author(s) declare no conflict of interest.

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