Journal of Information and Computing Science
J. Info. Comput. Sci., Vol. 20, No. 2, pp. 99-112

DOI: 10.4208/JICS-2025-006
December 2025

A Note on the Extremizers for a Nonlinear Hardy-Littlewood-Sobolev Inequality

Xingdong Tang[1,*] and Yang Zhang[1]

[1] *School of Mathematics and Statistics, Nanjing Univeristy of Information Science and Technology, Nanjing 210044, China.*

Abstract. The extremizers of a nonlinear Hardy-Littlewood-Sobolev inequality will be classified by making use of the Frank-Lieb argument, via the stereographic projection and spherical harmonic.

AMS subject classifications: 45G10, 35Q89

Key words: Funk-Hecke formula, Hardy-Littlewood-Sobolev inequality, Spherical harmonic functions, Stereographic projection.

1 Introduction

We consider the inequality

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f^p(x)g^p(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\le C(N,\lambda)\|\nabla f\|_2\|\nabla g\|_2, \tag{1.1}$$

where $N\ge 3$, $0<\lambda<N$, $p={(2N-\lambda)}/{(N-2)}$, f and g are non-negative real functions.

On the one hand, inequality (1.1) can be seen as a nonlinear generalization of the classical Hardy-Littlewood-Sobolev inequality

$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f(x)\,g(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\right|\le \pi^{\lambda/2}\frac{\Gamma((N-\lambda)/2)}{\Gamma(N-\lambda/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{1-\lambda/N}\|f\|_p\,\|g\|_p, \tag{1.2}$$

where $0<\lambda<N$, $p=2N/(2N-\lambda)$. On the other hand, inequality (1.1) is equivalent to the extremal problem

$$C(N,\lambda)=\sup\left\{\frac{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f^p(x)f^p(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}}{\|\nabla f\|_2^2}\,\middle|\, f\in \dot{H}^1\left(\mathbb{R}^N\right)\setminus\{0\},\ f(x)\ge 0,\text{a.e.}\,x\in\mathbb{R}^N\right\}. \tag{1.3}$$

*Corresponding author. *Email addresses:* txd@nuist.edu.cn (X. Tang), 202312150023@nuist.edu.cn (Y. Zhang)

By direct computations, we get that the Euler-Lagrange equation of (1.1) is (up to some constant) the nonlinear Hartree equation

$$\Delta u(x)+\int_{\mathbb{R}^N}\frac{f^p(y)}{|x-y|^\lambda}\mathrm{d}y f^{p-1}(x)=0,\quad x\in\mathbb{R}^N, \tag{1.4}$$

which appears in the mean field limit of quantum many-body system [1–5].

The main result of this paper is the following theorem:

Theorem 1.1. Let $N\geq 3$, $0<\lambda<N$, and $p=(2N-\lambda)/(N-2)$. Then for any non-negative $f,g\in \dot{H}^1(\mathbb{R}^N)\setminus\{0\}$, inequality (1.1) holds with the sharp constant

$$C(N,\lambda)=\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)(4\pi)^{\frac{N}{2}}}\right)\left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-\lambda}{2}\right)}{\Gamma(N)\Gamma\left(N-\frac{\lambda}{2}\right)}(4\pi)^N\right)^{\frac{N-2}{2N-\lambda}}. \tag{1.5}$$

Moreover, the equality in (1.1) holds if and only if

$$f(x)=\frac{c}{\left(1+\delta^2|x-x_0|^2\right)^{\frac{N-2}{2}}}\quad\text{and}\quad g(x)=\frac{c'}{\left(1+\delta^2|x-x_0|^2\right)^{\frac{N-2}{2}}},$$

where $c,c'>0$, $\delta>0$, and $x_0\in\mathbb{R}^N$.

In the elegant paper [6], via the stereographic projection, by making use of Riesz's rearrangement inequality, Lieb obtained the extremizers for (1.2). In [7], by making use of the stereographic projection and spherical harmonic, Frank and Lieb gave a rearrangement-free proof of existence of extremizers for (1.2). For the nonlinear Hardy-Littlewood-Sobolev inequality (1.1), by combining the Hardy-Littlewood-Sobolev and the sharp Sobolev inequality, the authors in [8, 9] obtained the extremizers for (1.1). In this paper, by making use of the argument of Frank and Lieb [7, 10], we give another proof of the existence of extremizers for (1.1).

The rest of this paper is organized as follows. In Section 2, we introduce some notations and the preliminary results about the stereographic projection and the Funk-Hecke formula of the spherical harmonic functions. In Section 3, we prove Theorem 1.1.

2 Notations and Preliminary Results

In this section, we introduce some notations and some preliminary results which will be used in the context. As usual, we write

$$\mathbb{R}^N:=\left\{x=(x_1,x_2,\cdots,x_N)\,\middle|\,x_j\in\mathbb{R},\quad 1\leq j\leq N\right\}$$

and denote the N-dimensional Euclidean space with the standard inner product and the Euclidean norm as

$$x\cdot y:=\sum_{j=1}^N x_jy_j,\quad\text{and}\quad |x|:=(x\cdot x)^{\frac{1}{2}},\quad\text{for all}\quad x,y\in\mathbb{R}^N.$$