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A Note on the Extremizers for a Nonlinear Hardy-Littlewood-Sobolev Inequality

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Abstract. The extremizers of a nonlinear Hardy-Littlewood-Sobolev inequality will be classified by making use of the Frank-Lieb argument, via the stereographic projection and spherical harmonic.

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Key words: Funk-Hecke formula, Hardy-Littlewood-Sobolev inequality, Spherical harmonic functions, Stereographic projection. †

1 Introduction

We consider the inequality

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f^p(x)g^p(y)}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\leq C(N,\lambda)\|\nabla f\|_2\|\nabla g\|_2, \tag{1.1}$$

where $N\geq 3$, $0<\lambda<N$, $p={}^{(2N-\lambda)}/{}_{(N-2)}$, f and g are non-negative real functions.

On the one hand, inequality (1.1) can be seen as a nonlinear generalization of the classical Hardy-Littlewood-Sobolev inequality

$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f(x)\,g(y)}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y\right|\leq \pi^{\lambda/2}\frac{\Gamma((N-\lambda)/2)}{\Gamma(N-\lambda/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{1-\lambda/N}\|f\|_p\,\|g\|_p, \tag{1.2}$$

where $0<\lambda<N$, $p=2N/(2N-\lambda)$. On the other hand, inequality (1.1) is equivalent to the extremal problem

$$C(N,\lambda)=\sup\left\{\frac{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f^p(x)f^p(y)}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}}{\|\nabla f\|_2^2}\,\middle|\, f\in\dot{H}^1\left(\mathbb{R}^N\right)\setminus\{0\},\, f(x)\geq 0,\text{a.e.}\,x\in\mathbb{R}^N\right\}. \tag{1.3}$$

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By direct computations, we get that the Euler-Lagrange equation of (1.1) is (up to some constant) the nonlinear Hartree equation

$$\Delta u(x)+\int_{\mathbb{R}^N}\frac{f^p(y)}{|x-y|^\lambda}\mathrm{d}y f^{p-1}(x)=0,\quad x\in\mathbb{R}^N, \tag{1.4}$$

which appears in the mean field limit of quantum many-body system [1–5].

The main result of this paper is the following theorem:

Theorem 1.1. Let $N\geq 3$, $0<\lambda<N$, and $p=(2N-\lambda)/(N-2)$. Then for any non-negative $f,g\in \dot{H}^1(\mathbb{R}^N)\setminus\{0\}$, inequality (1.1) holds with the sharp constant

$$C(N,\lambda)=\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)(4\pi)^{\frac{N}{2}}}\right)\left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-\lambda}{2}\right)}{\Gamma(N)\Gamma\left(N-\frac{\lambda}{2}\right)}(4\pi)^N\right)^{\frac{N-2}{2N-\lambda}}. \tag{1.5}$$

Moreover, the equality in (1.1) holds if and only if

$$f(x)=\frac{c}{\left(1+\delta^2|x-x_0|^2\right)^{\frac{N-2}{2}}}\quad\text{and}\quad g(x)=\frac{c'}{\left(1+\delta^2|x-x_0|^2\right)^{\frac{N-2}{2}}},$$

where $c,c'>0$, $\delta>0$, and $x_0\in\mathbb{R}^N$.

In the elegant paper [6], via the stereographic projection, by making use of Riesz's rearrangement inequality, Lieb obtained the extremizers for (1.2). In [7], by making use of the stereographic projection and spherical harmonic, Frank and Lieb gave a rearrangement-free proof of existence of extremizers for (1.2). For the nonlinear Hardy-Littlewood-Sobolev inequality (1.1), by combining the Hardy-Littlewood-Sobolev and the sharp Sobolev inequality, the authors in [8, 9] obtained the extremizers for (1.1). In this paper, by making use of the argument of Frank and Lieb [7,10], we give another proof of the existence of extremizers for (1.1).

The rest of this paper is organized as follows. In Section 2, we introduce some notations and the preliminary results about the stereographic projection and the Funk-Hecke formula of the spherical harmonic functions. In Section 3, we prove Theorem 1.1.

2 Notations and Preliminary Results

In this section, we introduce some notations and some preliminary results which will be used in the context. As usual, we write

$$\mathbb{R}^N:=\left\{x=(x_1,x_2,\cdots,x_N)\,\middle|\,x_j\in\mathbb{R},\quad 1\leq j\leq N\right\}$$

and denote the N-dimensional Euclidean space with the standard inner product and the Euclidean norm as

$$x\cdot y:=\sum_{j=1}^N x_j y_j,\quad\text{and}\quad |x|:=(x\cdot x)^{\frac{1}{2}},\quad\text{for all}\quad x,y\in\mathbb{R}^N.$$

In order to simplify the notation, we introduce the functions

$$\langle x\rangle=\left(1+|x|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \rho(x)=\left(\frac{2}{1+|x|^2}\right)^{\frac{1}{2}}.$$

We use $\mathbb{S}^N$ to stand for the unit sphere in $\mathbb{R}^{N+1}$, i.e.,

$$\mathbb{S}^N=\left\{\xi=(\xi_1,\xi_2,\cdots,\xi_{N+1})\in\mathbb{R}^{N+1}\,\middle|\,\sum_{j=1}^{N+1}\xi_j^2=1\right\},$$

and $g_{ij}(1\le i,j\le N+1)$ stand for the metric on $\mathbb{S}^N$, which is inherited from $\mathbb{R}^{N+1}$. In order to describe the functions both in $\mathbb{R}^N$ and on the sphere $\mathbb{S}^N$, let us introduce the stereographic projection and its inverse map, we denote the stereographic projection $\mathcal{S}:\mathbb{R}^N\mapsto\mathbb{S}^N\setminus\{(0,0,\cdots,0,-1)\}$ by

$$\mathcal{S}x=\left(\frac{2x}{1+|x|^2},\frac{1-|x|^2}{1+|x|^2}\right),$$

and its inverse maps $\mathcal{S}^{-1}:\mathbb{S}^N\setminus\{(0,0,\cdots,0,-1)\}\mapsto\mathbb{R}^N$ by

$$\mathcal{S}^{-1}(\xi_1,\xi_2,\cdots,\xi_{N+1})=\left(\frac{\xi_1}{1+\xi_{N+1}},\frac{\xi_2}{1+\xi_{N+1}},\cdots,\frac{\xi_N}{1+\xi_{N+1}}\right).$$

A simple calculation implies that

$$|\mathcal{S}x-\mathcal{S}y|=|x-y|\rho(x)\rho(y), \tag{2.1}$$

and $g_{ij}=\rho^4(x)\delta_{ij}$ under the stereographic projection (see [7,11]).

For any $f:\mathbb{R}^N\mapsto\mathbb{R}$, we denote $\mathcal{S}_*f:\mathbb{S}^N\setminus\{(0,0,\cdots,0,-1)\}\mapsto\mathbb{R}$ by

$$\mathcal{S}_*f(\xi)=\rho^{2-N}\left(\mathcal{S}^{-1}\xi\right)f\left(\mathcal{S}^{-1}\xi\right), \tag{2.2}$$

and for any $F:\mathbb{S}^N\setminus\{(0,0,\cdots,0,-1)\}\mapsto\mathbb{R}$, we denote $\mathcal{S}^*F:\mathbb{R}^N\mapsto\mathbb{R}$ by

$$\mathcal{S}^*F(x)=\rho^{N-2}(x)F(\mathcal{S}x). \tag{2.3}$$

For any $1\le p<+\infty$, let us denote by $L^p\left(\mathbb{R}^N\right)$ and $L^p\left(\mathbb{S}^N\right)$ the space of real-valued p-th power integrable functions on $\mathbb{R}^N$ and $\mathbb{S}^n$. Moreover, with a little abuse of notation, we equip $L^p\left(\mathbb{R}^N\right)$ and $L^p\left(\mathbb{S}^N\right)$ with the norms

$$\|f\|_p=\left(\int_{\mathbb{R}^N}|f(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}, \quad \text{for} \quad f\in L^p\left(\mathbb{R}^N\right)$$

and

$$\|F\|_p=\left(\int_{\mathbb{S}^N}|F(\xi)|^p\mathrm{d}\xi\right)^{\frac{1}{p}}, \quad \text{for} \quad F\in L^p\left(\mathbb{S}^N\right),$$

where $\mathrm{d}\xi$ is the standard volume element on the sphere $\mathbb{S}^N$, and can be expressed by the stereographic projection as

$$\mathrm{d}\xi = \rho^{2N}(x)\mathrm{d}x. \tag{2.4}$$

Hence, for any $F \in L^1(\mathbb{S}^N)$, we have the identity

$$\int_{\mathbb{S}^N} F(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^N} F(\mathcal{S}x)\rho^{2N}(x)\mathrm{d}x.$$

Let us denote by $\dot{H}^1(\mathbb{R}^N)$ the space of the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|f\|_{\dot{H}^1(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla f(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Also, we denote the $\dot{H}^1$ norm on the sphere $\mathbb{S}^N$ by

$$\|F\|_{\dot{H}^1(\mathbb{S}^N)} = \left(\int_{\mathbb{S}^N} |\nabla F(\xi)|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^N} \sum_{j=1}^N \left[\frac{\partial}{\partial x_j}(F\circ\mathcal{S})(x)\right]^2 \rho^{2N-4}(x)\mathrm{d}x\right)^{\frac{1}{2}}.$$

We denote the convolution integrals $\mathcal{I}_{\mathbb{R}^N}(f,g)$ in $\mathbb{R}^N$ and $\mathcal{I}_{\mathbb{S}^N}(F,G)$ on $\mathbb{S}^N$ respectively by

$$\mathcal{I}_{\mathbb{R}^N}(f,g) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y, \tag{2.5}$$

and

$$\mathcal{I}_{\mathbb{S}^N}(F,G) := \int_{\mathbb{S}^N}\int_{\mathbb{S}^N} \frac{F(\xi)G(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta. \tag{2.6}$$

Let us recall some basic results about the integral estimates (see [6,7,11]). The first one is that sharp Hardy-Littlewood-Sobolev inequality

$$\mathcal{I}_{\mathbb{R}^N}(f,g) \le \mathrm{I}(N,\lambda)\|f\|_{\frac{2N}{2N-\lambda}}\|g\|_{\frac{2N}{2N-\lambda}} \tag{2.7}$$

holds for any $f,g \in L^{\frac{2N}{2N-\lambda}}(\mathbb{R}^N)$, $\lambda \in (0,N)$ and

$$\mathrm{I}(N,\lambda) = \pi^{\frac{\lambda}{2}} \frac{\Gamma\left(\frac{N-\lambda}{2}\right)}{\Gamma\left(N-\frac{\lambda}{2}\right)} \left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{1-\frac{\lambda}{N}}. \tag{2.8}$$

The second one is that sharp Sobolev inequality

$$\|f\|_{\frac{2N}{N-2}} \le \mathrm{S}(N)\|\nabla f\|_2 \tag{2.9}$$

holds for any $f \in \dot{H}^1(\mathbb{R}^N)$, $N \ge 3$ and

$$\mathrm{S}(N) = \frac{4}{N(N-2)}\left(\frac{\Gamma\left(\frac{N+1}{2}\right)}{2\pi^{\frac{N+1}{2}}}\right)^{\frac{2}{N}}. \tag{2.10}$$

Let us denote

$$\mathrm{D}(N,\lambda)=\frac{\pi^{\frac{N}{2}}\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{N-\lambda}{2}\right)}\left(\frac{\Gamma\left(\frac{N-\lambda}{4}\right)}{\Gamma\left(\frac{N+\lambda}{4}\right)}\right)^2.$$

By the basic convolution identity from [6,11]

$$\frac{\mathrm{D}(N,\lambda)}{|x-y|^{\lambda}}=\int_{\mathbb{R}^N}\frac{1}{|x-z|^{\frac{N+\lambda}{2}}}\frac{1}{|z-y|^{\frac{N+\lambda}{2}}}\mathrm{d}z, \tag{2.11}$$

we can rewrite the integral $\mathcal{I}_{\mathbb{R}^N}(f,g)$ as

$$\mathcal{I}_{\mathbb{R}^N}(f,g)=\frac{1}{\mathrm{D}(N,\lambda)}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f(x)}{|x-z|^{\frac{N+\lambda}{2}}}\mathrm{d}x\right)\left(\int_{\mathbb{R}^N}\frac{g(x)}{|x-z|^{\frac{N+\lambda}{2}}}\mathrm{d}x\right)\mathrm{d}z. \tag{2.12}$$

In particular, we have

$$\mathcal{I}_{\mathbb{R}^N}(f,f)=\frac{1}{\mathrm{D}(N,\lambda)}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f(x)}{|x-z|^{\frac{N+\lambda}{2}}}\mathrm{d}x\right)^2\mathrm{d}z. \tag{2.13}$$

We now introduce the spherical harmonic functions, which is related to the spectral properties of the Laplace-Beltrami opertor for the sphere $\mathbb{S}^N$ (see [12, 13]). In fact, we have the orthogonal decomposition

$$L^2\left(\mathbb{S}^N\right)=\bigoplus_{k=0}^{\infty}\mathscr{H}_k^{N+1}, \tag{2.14}$$

where we use $\mathscr{H}_k^{N+1}(k\geq 0)$ to denote the mutually orthogonal subspace of the restriction on $\mathbb{S}^N$ of real harmonic polynomials, homogeneous of degree of k. The dimension of the subspace $\mathscr{H}_k^{N+1}$ can be computed as

$$\dim\mathscr{H}_k^{N+1}:=\begin{cases}1, & \text{if } k=0,\\ N+1, & \text{if } k=1,\\ \binom{k+N}{k}-\binom{k-2+N}{k-2}, & \text{if } k\geq 2.\end{cases}$$

We will use $\left\{Y_{k,j}\,|\,1\leq j\leq \dim\mathscr{H}_k^{N+1}\right\}$ to denote an orthonormal basis of $\mathscr{H}_k^{N+1}$. In particular, we have for the case $k=1$ that

$$Y_{1,j}(\xi)=\sqrt{\frac{(N+1)\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}}\xi_j,\quad 1\leq j\leq N+1,$$

and

$$\mathscr{H}_1^{N+1}=\mathrm{span}\left\{\xi_j\,\middle|\,1\leq j\leq N+1\right\}. \tag{2.15}$$

For the integer $k\geq 0$, let us denote

$$\mu_k(\lambda)=2^{N-\lambda}\pi^{\frac{N}{2}}\frac{\Gamma\left(k+\frac{\lambda}{2}\right)\Gamma\left(\frac{N-\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(k+N-\frac{\lambda}{2}\right)}. \tag{2.16}$$

The following Funk-Hecke formula of the spherical harmonic functions will be useful to calculate the convolution integrals on the sphere $\mathbb{S}^N$.

Lemma 2.1 ([12,13]). Let $\lambda\in(0,N)$ and $\mu_k(\lambda)$ be defined by (2.16), then for any $Y\in\mathscr{H}_k^{N+1}$, we have

$$\int_{\mathbb{S}^N}\frac{1}{|\xi-\eta|^{\lambda}}Y(\eta)\,\mathrm{d}\eta=\mu_k(\lambda)Y(\xi). \tag{2.17}$$

As a direct consequence of the above Funk-Hecke formula, we have the following result:

Lemma 2.2. Let $\lambda\in(0,N)$, and $\mu_k(\lambda)$ be defined by (2.16), then for any $Y\in\mathscr{H}_k^{N+1}$, we have

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\frac{1}{|\xi-\eta|^{N-2}}\frac{1}{|\eta-\sigma|^{\lambda}}Y(\sigma)\,\mathrm{d}\sigma\mathrm{d}\eta=\mu_k(N-2)\mu_k(\lambda)Y(\xi),$$
$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\frac{1}{|\xi-\eta|^{N-2}}\frac{1}{|\eta-\sigma|^{\lambda}}Y(\eta)\,\mathrm{d}\eta\mathrm{d}\sigma=\mu_k(N-2)\mu_0(\lambda)Y(\xi).$$

3 Proof of Theorem 1.1

Let us rewrite (1.1) as

$$\mathrm{C}(N,\lambda)=\sup\left\{\frac{\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)^{\frac{1}{p}}}{\|\nabla f\|_2\|\nabla g\|_2}\,\middle|\,\begin{array}{c}f,\,g\in\dot{H}^1\left(\mathbb{R}^N\right)\setminus\{0\},\\ f(x),\,g(x)\geq 0,\text{ a.e. }x\in\mathbb{R}^n\end{array}\right\}. \tag{3.1}$$

Lemma 3.1. The sharp constant $\mathrm{C}(N,\lambda)$ in (3.1) is finite.

Proof. By the Hardy-Littlewood-Sobolev inequality (2.7) and the Sobolev inequality (2.9), we have

$$\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)\leq I(N,\lambda)\|f\|^p_{\frac{2N}{N-2}}\|g\|^p_{\frac{2N}{N-2}}\leq I(N,\lambda)S(N)^{2p}\|\nabla f\|_2^p\|\nabla g\|_2^p.$$

Therefore, we have $\mathrm{C}(N,\lambda)\leq I(N,\lambda)^{\frac{1}{p}}S(N)^2$. This ends the proof of Lemma 3.1. □

Next, we show some basic facts related to the extremizers of (3.1).

Lemma 3.2. Let $\mathrm{C}(N,\lambda)$ be given by (3.1), and $\mathcal{I}_{\mathbb{R}^N}$ be defined by (2.5). If the non-negative pair $(f,g)\in\dot{H}^1\left(\mathbb{R}^N\right)\times\dot{H}^1\left(\mathbb{R}^N\right)$ satisfies

$$\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)^{\frac{1}{p}}=\mathrm{C}(N,\lambda)\|\nabla f\|_2\|\nabla g\|_2, \tag{3.2}$$

then there exists $\beta\in\mathbb{R}\setminus\{0\}$ such that

$$f(x)=\beta g(x)\quad\text{a.e.}\quad x\in\mathbb{R}^N. \tag{3.3}$$

Proof. We argue by contradiction. Let the pair $(f,g)\in \dot{H}^1(\mathbb{R}^N)\times \dot{H}^1(\mathbb{R}^N)$ satisfy (3.2) and suppose that there does not exist $\beta\in\mathbb{R}\setminus\{0\}$ such that,

$$f(x)=\beta g(x) \quad \text{a.e.} \quad x\in\mathbb{R}^N.$$

First, by (2.12), we can rewrite $\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)$ as

$$\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)=\frac{1}{\mathrm{D}(N,\lambda)}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f^p(x)}{|x-z|^{\frac{N+\lambda}{2}}}\mathrm{d}x\right)\left(\int_{\mathbb{R}^N}\frac{g^p(x)}{|x-z|^{\frac{N+\lambda}{2}}}\mathrm{d}x\right)\mathrm{d}z. \tag{3.4}$$

By the Riesz theory in [11] and the Cauchy-Schwarz inequality, we obtain from our assumptions (3.4) that

$$\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)<\frac{1}{\mathrm{D}(N,\lambda)}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f^p(x)}{|x-z|^{\frac{N+\lambda}{2}}}\mathrm{d}x\right)^2\mathrm{d}z\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{g^p(x)}{|x-z|^{\frac{N+\lambda}{2}}}\mathrm{d}x\right)^2\mathrm{d}z\right)^{\frac{1}{2}},$$

which together with (2.13) implies that

$$\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)<\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)^{\frac{1}{2}}\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)^{\frac{1}{2}}. \tag{3.5}$$

Hence, by inserting (3.1) into (3.5), we get

$$\mathcal{I}_{\mathbb{R}^N}(f^p,g^p)^{\frac{1}{p}}<\mathrm{C}(N,\lambda)\|\nabla f\|_2\|\nabla g\|_2,$$

which contradicts with (3.2). This ends the proof of Lemma 3.2. □

Based on the basic fact in (2.13), we give an alternative characterization of sharp constant $\mathrm{C}(N,\lambda)$ in (3.1).

Lemma 3.3. Let $\mathcal{I}_{\mathbb{R}^N}$ be defined by (2.5), then we have

$$\mathrm{C}(N,\lambda)=\sup\left\{\frac{\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)^{\frac{1}{p}}}{\|\nabla f\|_2^2}\;\middle|\;\begin{array}{l} f\in\dot{H}^1\left(\mathbb{R}^N\right)\setminus\{0\},\\ f(x)\geq 0,\ \text{a.e. } x\in\mathbb{R}^n\end{array}\right\}. \tag{3.6}$$

Proof. Let us denote

$$m(N,\lambda)=\sup\left\{\frac{\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)^{\frac{1}{p}}}{\|\nabla f\|_2^2}\;\middle|\;\begin{array}{l} f\in\dot{H}^1\left(\mathbb{R}^N\right)\setminus\{0\},\\ f(x)\geq 0,\ \text{a.e. } x\in\mathbb{R}^n\end{array}\right\}.$$

On the one hand, by taking $g=f$ in (3.1), we have

$$\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)^{\frac{1}{p}}\leq\mathrm{C}(N,\lambda)\|\nabla f\|_2^2, \quad \text{for any} \quad f\in\dot{H}^1\left(\mathbb{R}^N\right)\setminus\{0\},$$

which implies that,

$$m(N,\lambda)\leq\mathrm{C}(N,\lambda). \tag{3.7}$$

On the other hand, by use of (2.12), (2.13) and the Cauchy-Schwarz inequality, we obtain that

$$\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)\le\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)^{\frac{1}{2}}\mathcal{I}_{\mathbb{R}^N}(g^p,g^p)^{\frac{1}{2}}\le m(N,\lambda)^p\|\nabla f\|_2^p\|\nabla g\|_2^p,$$

therefore we have

$$\mathrm{C}(N,\lambda)\le m(N,\lambda).$$

which together with (3.7) implies the result. □

In orde to make use of sharp Hardy-Littlewood-Sobolev inequality on the sphere $\mathbb{S}^N$ to prove the results in Theorem 1.1, we will transform the integrals and estimates on $\mathbb{R}^N$ to those on $\mathbb{S}^N$ by making use of the stereographic projection, and give some estimates on $\mathbb{S}^N$ from [7].

By making use of the stereographic projection, we have the following result.

Lemma 3.4. Let $\mathcal{I}_{\mathbb{R}^N}$ and $\mathcal{I}_{\mathbb{S}^N}$ be defined by (2.5) and (2.6) respectively and $\mathcal{S}_*f$ be defined by (2.2), then for any $f\in H^1(\mathbb{R}^N)\setminus\{0\}$, we have

$$\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)=\mathcal{I}_{\mathbb{S}^N}\left((\mathcal{S}_*f)^p,(\mathcal{S}_*f)^p\right),$$
$$\|\nabla f\|_2^2=\mathcal{E}(\mathcal{S}_*f),$$

where

$$\mathcal{E}(F)=\|F\|^2_{H^1(\mathbb{S}^N)}+\frac{N(N-2)}{4}\int_{\mathbb{S}^N}|F(\xi)|^2\mathrm{d}\xi. \tag{3.8}$$

Proof. By (2.1) and $p={}^{(2N-\lambda)}/{}_{(N-2)}$, we have

$$\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left(\rho^{2-N}(x)f(x)\right)^p\left(\rho^{2-N}(y)f(y)\right)^p}{|\mathcal{S}x-\mathcal{S}y|^\lambda}\rho^{2N}(x)\rho^{2N}(y)\mathrm{d}x\mathrm{d}y,$$

which, together with (2.2), implies that

$$\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\mathcal{S}_*f)^p(\xi)(\mathcal{S}_*f)^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta.$$

Next, by an elementary calculation (see for instance Appendix A in [7]), we have

$$\|\nabla f\|_2^2=\mathcal{E}(\mathcal{S}_*f).$$

This finishes the proof of Lemma 3.4. □

Now, we give two useful results related to spherical harmonics.

Lemma 3.5. [7] Let $\lambda\in(0,N)$ and $F:\mathbb{S}^N\mapsto\mathbb{R}$, then we have

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\frac{\xi\cdot\eta}{|\xi-\eta|^\lambda}F(\xi)F(\eta)\mathrm{d}\xi\mathrm{d}\eta\ge\frac{\lambda}{2N-\lambda}\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\frac{1}{|\xi-\eta|^\lambda}F(\xi)F(\eta)\mathrm{d}\xi\mathrm{d}\eta$$

with equality if and only if F is constant.

Lemma 3.6. [7] Let $\mathcal{E}$ be defined by (3.8). For any $F\in H^1\left(\mathbb{S}^N\right)$, we have

$$\sum_{j=1}^{N+1}\mathcal{E}\left(F_j\right)=\mathcal{E}(F)+N\int_{\mathbb{S}^N}|F(\xi)|^2\mathrm{d}\xi,$$

where $F_j(\xi)=\xi_jF(\xi),j=1,2,\cdots,N+1$.

Along the line in [7], we introduce a useful lemma, which allows us to normalize functions with vanishing center of mass.

Lemma 3.7. [7] Let $F\in L^1\left(\mathbb{S}^N\right)$ with $\int_{\mathbb{S}^N}F(\xi)\mathrm{d}\xi\neq 0$. Then there exist a constant $\delta\in(0,1)$ and an orthogonal $(N+1)\times(N+1)$ matrix Q such that,

$$\int_{\mathbb{S}^N}Q^{\intercal}\mathcal{S}\circ\mathcal{D}_\delta\circ\mathcal{S}^{-1}Q\xi F(\xi)\mathrm{d}\xi=0,$$

where $\mathcal{D}_\delta x=\delta x$.

The above lemma can be generalized to the following result.

Lemma 3.8. Let $\mathcal{I}_{\mathbb{S}^N}$ and $\mathcal{E}$ be defined by (2.6) and (3.8) respectively. For any $F:\mathbb{S}^N\mapsto\mathbb{R}$ with $\mathcal{I}_{\mathbb{S}^N}(F^p,F^p)<+\infty$, $\mathcal{E}(F)<+\infty$ and

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\xi\frac{F^p(\xi)F^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta\neq 0,$$

there exist a constant $\delta\in(0,1)$ and an orthogonal $(N+1)\times(N+1)$ matrix Q, such that

$$F_{Q,\delta}(\xi):=\left(\frac{\rho\left(\mathcal{D}_{\frac{1}{\delta}}\circ\mathcal{S}^{-1}Q\xi\right)}{\delta\rho(\mathcal{S}^{-1}Q\xi)}\right)^{N-2}F\left(Q^{\intercal}\mathcal{S}\circ\mathcal{D}_{\frac{1}{\delta}}\circ\mathcal{S}^{-1}Q\xi\right) \tag{3.9}$$

satisfies $\mathcal{I}_{\mathbb{S}^N}\left(F^p_{Q,\delta},F^p_{Q,\delta}\right)=\mathcal{I}_{\mathbb{S}^N}(F^p,F^p)$, $\mathcal{E}(F_{Q,\delta})=\mathcal{E}(F)$ and

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\xi\frac{F^p_{Q,\delta}(\xi)F^p_{Q,\delta}(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta=0. \tag{3.10}$$

Proof. Since $\displaystyle\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\frac{F^p(\xi)F^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta<+\infty$, the function $G(\xi)=\displaystyle\int_{\mathbb{S}^N}\frac{F^p(\xi)F^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\eta$ belongs to $L^1\left(\mathbb{S}^N\right)$. By Lemma 3.7, there exist a constant $\delta\in(0,1)$ and an orthogonal $(N+1)\times(N+1)$ matrix Q such that,

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}Q^{\intercal}\mathcal{S}\circ\mathcal{D}_\delta\circ\mathcal{S}^{-1}Q\xi\frac{F^p(\xi)F^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta=0.$$

Since the functionals $\mathcal{I}_{\mathbb{S}^N}$ and $\mathcal{E}$ defined by (2.6) and (3.8) are invariant under the orthogonal transformations, we have $\mathcal{I}_{\mathbb{S}^N}\left((F\circ Q^\intercal)^p,(F\circ Q^\intercal)^p\right)=\mathcal{I}_{\mathbb{S}^N}(F^p,F^p)$, $\mathcal{E}(F\circ Q^\intercal)=\mathcal{E}(F)$ and

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N} Q^\intercal\mathcal{S}\circ\mathcal{D}_\delta\circ\mathcal{S}^{-1}\xi\frac{(F\circ Q^\intercal)^p(\xi)(F\circ Q^\intercal)^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta=0. \tag{3.11}$$

Let $\tilde{f}(x)=[\mathcal{S}^*(F\circ Q^\intercal)](x)$. By (3.11), (2.1) and Lemma 3.4, we have $\mathcal{I}_{\mathbb{R}^N}\left(\tilde{f}^p,\tilde{f}^p\right)=\mathcal{I}_{\mathbb{S}^N}\left((F\circ Q^\intercal)^p,(F\circ Q^\intercal)^p\right)$, $\|\nabla\tilde{f}\|_2^2=\mathcal{E}\left(\tilde{F}\right)$ and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} Q^\intercal\mathcal{S}\circ\mathcal{D}_\delta x\frac{\tilde{f}^p(x)\tilde{f}^p(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y=0. \tag{3.12}$$

Let $\tilde{f}_\delta(x)=\frac{1}{\delta^{(N-2)/2}}\tilde{f}\left(\frac{x}{\delta}\right)$. By the scaling invariance of the functionals $\mathcal{I}_{\mathbb{R}^N}$ and the $\dot{H}^1$-norm, we have $\mathcal{I}_{\mathbb{R}^N}\left(\tilde{f}_\delta^p,\tilde{f}_\delta^p\right)=\mathcal{I}_{\mathbb{R}^N}\left(\tilde{f}^p,\tilde{f}^p\right)$, $\|\nabla\tilde{f}\|_2^2=\|\nabla\tilde{f}\|_2^2$ and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} Q^\intercal\mathcal{S}x\frac{\tilde{f}_\delta^p(x)\tilde{f}_\delta^p(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y=0. \tag{3.13}$$

Let $\tilde{F}_\delta(\xi)=\mathcal{S}_*\tilde{f}_\delta(\xi)$. By use of (2.1) and Lemma 3.4 again, we have $\mathcal{I}_{\mathbb{S}^N}\left(\tilde{F}_\delta^p,\tilde{F}_\delta^p\right)=\mathcal{I}_{\mathbb{R}^N}\left(\tilde{f}_\delta^p,\tilde{f}_\delta^p\right)$, $\mathcal{E}\left(\tilde{F}_\delta\right)=\|\nabla\tilde{f}_\delta\|_2^2$ and

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N} Q^\intercal\xi\frac{\tilde{F}_\delta^p(\xi)\tilde{F}_\delta^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta=0. \tag{3.14}$$

By the invariance of the functionals $\mathcal{I}_{\mathbb{S}^N}$ and $\mathcal{E}$ under the orthogonal transforms once again, we obtain $\mathcal{I}_{\mathbb{S}^N}\left(\left(\tilde{F}_\delta\circ Q\right)^p,\left(\tilde{F}_\delta\circ Q\right)^p\right)=\mathcal{I}_{\mathbb{S}^N}\left(\tilde{F}_\delta^p,\tilde{F}_\delta^p\right)$, $\mathcal{E}\left(\tilde{F}_\delta\circ Q\right)=\mathcal{E}\left(\tilde{F}_\delta\right)$ and

$$\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\xi\frac{\left(\tilde{F}_\delta\circ Q\right)^p(\xi)\left(\tilde{F}_\delta\circ Q\right)^p(\eta)}{|\xi-\eta|^\lambda}\mathrm{d}\xi\mathrm{d}\eta=0. \tag{3.15}$$

Finally, combining (3.11), (3.12), (3.13), (3.14) with (3.15), we can obtain the result, and complete the proof of Lemma 3.8. □

The following lemma shows the rigidity result that the extremizer with vanishing center must be a constant.

Lemma 3.9. Let $\mathcal{I}_{\mathbb{S}^N}$ be defined by (2.6), and $U_j(\xi)=\xi_jU(\xi)$ $(1\le j\le N+1)$. If U is an extremizer of (3.6) with

$$\mathcal{I}_{\mathbb{S}^N}\left(U^p,U^{p-1}U_j\right)=0,\quad 1\le j\le N+1,$$

then there exists $c>0$ such that $U(\xi)=c$, a.e $\xi\in\mathbb{S}^N$.

Proof. Suppose that $U\in H^1(\mathbb{S}^N)\setminus\{0\}$ is an extremizer of (3.6) with

$$\mathcal{I}_{\mathbb{S}^N}\left(U^p,U^{p-1}U_j\right)=0,\quad 1\le j\le N+1, \tag{3.16}$$

and

$$\mathcal{Q}(U)=\sup\left\{\mathcal{Q}(F)\,|\,F\in H^1\left(\mathbb{S}^N\right)\setminus\{0\}\right\}, \tag{3.17}$$

where $\mathcal{Q}(F)=\frac{\mathcal{I}_{\mathbb{S}^N}(F^p,F^p)^{\frac{1}{p}}}{\mathcal{E}(F)}$. It follows for any $F\in H^1(\mathbb{S}^N)$ that

$$\mathcal{Q}'(U)(F)=0,\quad\text{and}\quad \mathcal{Q}''(U)(F,F)\le 0, \tag{3.18}$$

For any $F\in H^1(\mathbb{S}^N)$, by an elementary calculation, (3.18) gives

$$\mathcal{I}_{\mathbb{S}^N}(U^p,U^p)\mathcal{E}'(U)(F)=2\mathcal{E}(U)\mathcal{I}_{\mathbb{S}^N}\left(U^p,U^{p-1}F\right)$$

and

$$\begin{aligned}\mathcal{I}_{\mathbb{S}^N}(U^p,U^p)\mathcal{E}''(U)(F,F)&+4(p-1)\frac{\mathcal{I}_{\mathbb{S}^N}(U^p,U^p)}{\mathcal{E}(U)}\left[\mathcal{I}_{\mathbb{S}^N}\left(U^p,U^{p-1}F\right)\right]^2\\ &\ge 2p\mathcal{E}(U)\mathcal{I}_{\mathbb{S}^N}\left(U^{p-1}F,U^{p-1}F\right)+2(p-1)\mathcal{E}(U)\mathcal{I}_{\mathbb{S}^N}\left(U^p,U^{p-2}F^2\right),\end{aligned}$$

which, together with (3.16), implies for $F=U_j$ that

$$\begin{aligned}&\mathcal{I}_{\mathbb{S}^N}(U^p,U^p)\sum_{j=1}^{N+1}\mathcal{E}''(U)(U_j,U_j)\\ &\ge 2p\mathcal{E}(U)\sum_{j=1}^{N+1}\mathcal{I}_{\mathbb{S}^N}\left(U^{p-1}U_j,U^{p-1}U_j\right)+2(p-1)\mathcal{E}(U)\sum_{j=1}^{N+1}\mathcal{I}_{\mathbb{S}^N}\left(U^p,U^{p-2}U_j^2\right).\end{aligned} \tag{3.19}$$

The left hand side of (3.19) can be estimated by Lemma 3.6 that

$$\sum_{j=1}^{N+1}\mathcal{E}''(U)(U_j,U_j)=2\mathcal{E}(U)+2N\int_{\mathbb{S}^N}U^2(\xi)\,\mathrm{d}\xi. \tag{3.20}$$

The first term in the right hand side of (3.19) can be estimated by Lemma 3.5 that

$$\sum_{j=1}^{N+1}\mathcal{I}_{\mathbb{S}^N}\left(U^{p-1}U_j,U^{p-1}U_j\right)\ge\frac{\lambda}{2N-\lambda}\mathcal{I}_{\mathbb{S}^N}(U^p,U^p). \tag{3.21}$$

and the second term in the right hand side of (3.19) as

$$\sum_{j=1}^{N+1}\mathcal{I}_{\mathbb{S}^N}\left(U^p,U^{p-2}U_j^2\right)=\mathcal{I}_{\mathbb{S}^N}(U^p,U^p). \tag{3.22}$$

By inserting (3.20), (3.21) and (3.22) into (3.19), we get

$$\mathcal{E}(U)+N\int_{\mathbb{S}^N}U^2(\xi)\mathrm{d}\xi \geq \frac{N+2}{N-2}\mathcal{E}(U). \tag{3.23}$$

where we use the fact that $p=\frac{2N-\lambda}{N-2}$. The above estimate implies that

$$\frac{4}{N-2}\|U\|^2_{\dot{H}^1(\mathbb{S}^N)}\leq 0,$$

which, together with the fact that $\int_{\mathbb{S}^N}U^2(\xi)\mathrm{d}\xi>0$, implies that there exists $c\in\mathbb{R}\setminus\{0\}$ such that

$$U(\xi)=c, \quad \text{a.e.} \quad \xi\in\mathbb{S}^N.$$

This completes the proof of Lemma 3.9. □

As a consequence of Lemma 3.8 and Lemma 3.9, we have

Corollary 3.1. Let $\mathrm{C}(N,\lambda)$, $\mathcal{I}_{\mathbb{S}^N}$ and $\mathcal{E}$ be defined by (1.1) , (2.6) and (3.8) respectively. The equality

$$\mathrm{C}(N,\lambda)=\frac{\mathcal{I}_{\mathbb{S}^N}(U^p,U^p)^{\frac{1}{p}}}{\mathcal{E}(U)} \tag{3.24}$$

holds if and only if

$$U(\xi)=\frac{c}{(1-\omega\cdot\xi)^{N-2}}, \quad \text{for some} \quad c>0, \text{ and } \omega\in\mathbb{R}^{N+1} \text{ with } |\omega|<1.$$

Moreover, the constant $\mathrm{C}(N,\lambda)$ can be calculated explicitly as follows:

$$\mathrm{C}(N,\lambda)=\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)(4\pi)^{\frac{N}{2}}}\right)\left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-\lambda}{2}\right)}{\Gamma(N)\Gamma\left(N-\frac{\lambda}{2}\right)}(4\pi)^N\right)^{\frac{N-2}{2N-\lambda}}. \tag{3.25}$$

Proof. It suffices to show that (3.25) holds. Indeed, by taking $U(\xi)=c$ in (3.24), Lemma 2.1 and the fact that $\int_{\mathbb{S}^N}\mathrm{d}\xi=\frac{\Gamma\left(\frac{N}{2}\right)(4\pi)^{\frac{N}{2}}}{\Gamma(N)}$, we obtain the result. □

Proof of Theorem 1.1. It suffices to show the extremizer. By combining Lemma 3.3, Lemma 3.4 and Corollary 3.1, we obtain that

$$\mathrm{C}(N,\lambda)=\frac{\mathcal{I}_{\mathbb{R}^N}(f^p,f^p)^{\frac{1}{p}}}{\|\nabla f\|_2^2} \tag{3.26}$$

if and only if

$$f(x)=\frac{c}{\left(1+\delta^2|x-x_0|^2\right)^{\frac{N-2}{2}}}, \quad \text{for some} \quad c\in\mathbb{R}\setminus\{0\},\ \delta>0, \text{ and } x_0\in\mathbb{R}^N.$$

Therefore by Lemma 3.2, we obtain that the equality

$$C(N,\lambda)=\frac{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f^p(x)g^p(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}}{\|\nabla f\|_2\|\nabla g\|_2}$$

holds if and only if

$$f(x)=\frac{c}{\left(1+\delta^2|x-x_0|^2\right)^{\frac{N-2}{2}}},\quad \text{and}\quad g(x)=\frac{c'}{\left(1+\delta^2|x-x_0|^2\right)^{\frac{N-2}{2}}},$$

for some $c,c'>0$, $\delta>0$, and $x_0\in\mathbb{R}^N$. This completes the proof of Theorem 1.1. □

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Conflicts of Interest

The authors declare no conflict of interest.

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