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A Note on the Determinant of a Special Class of Q-Walk Matrices

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Abstract. For a graph G of order n, its Q-walk matrix is defined by $W_Q(G) = [e,Qe,\cdots,Q^{n-1}e]$, where Q is the signless Laplacian matrix of G and e denotes the all-one column vector. Let $G\circ P_k$ represent the rooted product graph of G and a path P_k. In this note, we establish the relationship between determinants of $W_Q(G)$ and $W_Q(G\circ P_k)$ for $k=2,3$.

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Key words: Signless Laplacian matrix, Q-walk matrix, rooted product graph, determinant.

1 Introduction

All graphs considered are simple, that is, finite undirected graphs without multiple edges and loops. For a graph G of order n, let $A(G)$ and $D(G)$ denote its adjacency matrix and degree diagonal matrix, respectively. The matrix $Q(G)=D(G)+A(G)$ is called the signless Laplacian matrix of G (Q-matrix for short). All the eigenvalues of $Q(G)$ are called the Q-eigenvalues of G. Without causing confusion, we use A and Q instead of $A(G)$ and $Q(G)$, respectively. The A-walk matrix (resp. Q-walk matrix) of G is defined by $W_A(G)=[e,Ae,\cdots,A^{n-1}e]$ (resp. $W_Q(G)=[e,Qe,\cdots,Q^{n-1}e]$), where e represents the all-one column vector.

The walk matrices of graphs not only possess some fascinating properties, but also play a crucial role in the problem of characterizing families of graphs determined by generalized spectra. For example, it was proved in [6,7] that the determinant of A-walk matrix of an n-vertex graph G must be a multiple of $2^{\lfloor \frac{n}{2} \rfloor}$. Similarly, the determinant of its Q-walk matrix must be a multiple of $2^{\lfloor \frac{3n-2}{2} \rfloor}$ (see [5]). As a result, Wang [6] gave a

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simple arithmetic criterion for graphs being determined by their generalized spectra in terms of the determinants of A-walk matrices. Subsequently, Qiu et al. [5] presented a counterpart for graphs being determined by their generalized Q-spectra in terms of the determinants of Q-walk matrices. For more results about this topic, also see [8,9].

It is well-known that the graphs being determined by their generalized spectra are very few and have special structural properties. Mao et al. [3] first used the root product operation of graphs to construct the graphs being determined by their generalized spectra. Given a graph G of order n and a graph H with a root vertex u, then the rooted product graph of G and H is obtained by copying one graph G and copying n graphs H, by gluing the i-th vertex of G and the rooted vertex u in the i-th copy of H for $1\leq i\leq n$ (see Godsil and McKay [1]). Let P_k be the rooted path of order k and the root vertex be an endpoint, then the rooted product $G\circ P_k$ of graph G and P_k is described in Figure 1. In 2015, Mao et al. [3] gave a sufficient condition for the rooted product $G\circ P_k$ with $k=2$ being determined by their generalized spectra. In 2023, Mao and Wang [4] considered the same problem for the case of $k=3$ and 4. In the meantime, they proposed the following conjecture about the relationship between $\det(W_A(G))$ and $\det(W_A(G\circ P_k))$.

Conjecture 1.1 ([4]). *For the rooted product graph $G\circ P_k$ with $k\geq 2$,*

$$\det(W_A(G\circ P_k))=\pm a_0^{\lfloor \frac{k}{2}\rfloor}(\det(W_A(G)))^k,$$

where a_0 is the constant term with respect to the characteristic polynomial of $A(G)$.

Remark that Conjecture 1.1 is true for $k=2$ in [3] and $k=3,4$ in [4]. Recently, Wang et al. [10] proved the conjecture above completely. Now if we replace the A-walk matrix $W_A(G)$ with the Q-walk matrix $W_Q(G)$ of G, then a natural problem arises as follows:

Problem 1.1. Determine the relationship between $\det(W_Q(G))$ and $\det(W_Q(G\circ P_k))$.

In this note, we investigate the above problem and the following determinant relationships between $W_Q(G)$ and $W_Q(G\circ P_k)$ are presented for $k=2,3$.

Theorem 1.1. *For the rooted product graph $G\circ P_k$ with $k=2,3$, we have*

$$\det(W_Q(G\circ P_2))=\pm a_0(\det(W_Q(G)))^2,\ \det(W_Q(G\circ P_3))=\pm a_0^2(\det(W_Q(G)))^3,$$

where a_0 is the constant term with respect to the characteristic polynomial of the matrix $Q(G)$.