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Common Fixed Point Results for a Pair of Generalized Nonlinear Mappings in Complete Metric Spaces

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Abstract. In this paper, under the asymptotically regular condition, we investigate the existence of a unique common fixed point for a pair of generalized nonlinear mappings in the framework of complete metric spaces. Additionally, we extend our analysis to the case involving two control functions. Our work generalizes some results in recent papers.

AMS subject classifications: 47H10, 47H17

Key words: A pair of generalized nonlinear mappings, Asymptotically regular, Control functions, Metric spaces.

1 Introduction

Let (X,d) be a real complete metric space and $T,S:X\to X$ be two self-mappings. We denote the set of fixed points of T as $F(T)=\{x\in X:Tx=x\}$. Two self-mappings T and S are said to be a pair of generalized nonexpansive mappings if there exist three real numbers $M,K\in[0,+\infty)$ and $C\in[0,+\infty)$ such that

$$d(Tx,Sy)\le Md(x,y)+K[d(x,Tx)+d(y,Sy)]+C[d(x,Sy)+d(y,Tx)] \tag{1.1}$$

for all $x,y\in X$.

The concept of a pair of generalized nonexpansiveness is more general than that of contraction or nonexpansiveness [1-2]. In 1973, Hardy and Rogers [3] demonstrated the foundational significance of this concept, they revealed that if

$$M+2K+2C<1 \qquad \text{and} \qquad T=S,$$

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then the Picard iterative algorithm: select $x_0 \in X$,

$$x_{n+1} = Tx_n, \quad n \geq 0 \tag{1.2}$$

converges strongly to a unique fixed point of T in X. Under conditions

$$M+2K+2C \leq 1 \qquad \text{and} \qquad T=S,$$

Shimi [4] showed that the Krasnoselskii iterative algorithm: select $x_0 \in X$,

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), \qquad n \geq 0 \tag{1.3}$$

converges strongly to a fixed point of T in uniformly convex Banach spaces. The common fixed point problem of operators has significant theoretical value and applications. For a pair of mappings T and S, Bose [5] considered the case of

$$M+2K+2C \leq 1 \qquad \text{and} \qquad C \neq 0,$$

he obtained T and S have a common fixed point in uniformly convex Banach spaces. Based on the above work, Wang and Zhang [6] studied some sufficient conditions for a pair of nonlinear mappings (which include many contractive and expansive type mappings) in convex metric spaces, their analysis focused on the condition $(k,M,K,C) \in \Gamma_\rho$. Furthmore, Wang and Fan [7] used the Krasnoselskii-Mann iterative algorithm of T and S to approximate a unique common fixed point in uniformly convex metric spaces under certain specified conditions:

$$M+2K+2C \leq 1 \qquad \text{and} \qquad K,C \neq 0.$$

Without the need for the strict condition $M+2K+2C \leq 1$, suppose

$$T=S, \qquad M<1 \qquad \text{and} \qquad C=0,$$

Bisht [8] proved that T has a unique fixed point and the Picard iterative algorithm of T converges strongly to the unique fixed point under the asymptotic regularity of T and some weakly continuous conditions. Inspired by the work of [8], Gornicki[9] replaced M in (1.1) by some control functions and obtained some fixed point results.

Very recently, Khan etc.[10] extended the main results of [8] and gave the following result (the case of a pair of mappings).

Theorem 1.1. *Suppose (X,d) is a complete metric space, T and S are asymptotically regular on X satisfying (1.1) with the conditions $M<1$ and $C=0$. Let T and S be either k-continuous or orbitally continuous. Then T and S admit a unique fixed point p. Moreover, Picard iterative algorithms $\{T^n x\}$ and $\{S^n x\}$ both converge strongly to p.*