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Generalized Estimation of Numerical Radius for the off-Diagonal 2×2 Operator Matrices and Its Application

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Abstract. We give some generalized upper bounds for the numerical radius of off-diagonal 2×2 operator matrices. These inequalities are mainly based on the extension Buzano inequality and the generalized Young inequality. And our bounds refine and generalize the existing related upper bounds. Moreover, the conclusion is applied to the non-monic operator polynomials and gives a new bound for the eigenvalues of these operator polynomials.

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Key words: Numerical radius, operator matrix, extension Buzano inequality, Young inequality.

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1 Introduction

Let $\mathcal{B}(H)$ be the space of all bounded linear operators on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. An operator $A\in\mathcal{B}(H)$ is called positive if $\langle Ax,x\rangle\geq 0$ for all $x\in H$. An operator $A\in\mathcal{B}(H)$ is called uniformly hyponormal operator if $A^*A-AA^*\geq M>0$. The usual operator norm, the numerical radius of $A\in\mathcal{B}(H)$ are defined by

$$\|A\|=\sup\{|\langle Ax,y\rangle|: x,y\in H,\|x\|=\|y\|=1\},$$
$$\omega(A)=\sup\{|\langle Ax,x\rangle|: x\in H,\|x\|=1\}.$$

It has been shown that $\omega(A)$ defines a norm on $\mathcal{B}(H)$. And the above two norms are equivalent, i.e., for every $A\in\mathcal{B}(H)$,

$$\frac{1}{2}\|A\|\leq\omega(A)\leq\|A\|. \tag{1.1}$$

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If $R(A) \perp R(A^*)$, then the first inequality in (1.1) becomes an equality. The equality of the second inequality in (1.1) holds if and only if A is normaloid (i.e., $r(A) = \|A\|$, where $r(A)$ is the spectral radius of $A \in \mathcal{B}(H)$).

In [16] and [17], Kittaneh improved the inequality in (1.1). It has been shown that

$$\omega(A) \le \frac{1}{2}\||A|+|A^*|\| \le \frac{1}{2}(\|A\|+\|A^2\|^{1/2}), \tag{1.2}$$

where $|A| = (A^*A)^{1/2}$ is the absolute value of $A \in \mathcal{B}(H)$, and

$$\frac{1}{4}\|A^*A+AA^*\| \le \omega^2(A) \le \frac{1}{2}\|A^*A+AA^*\|. \tag{1.3}$$

In [4], Bhunia and Paul refines the right hand inequality in (1.3). It is well known that

$$\omega^2(A) \le \frac{1}{4}\||A|^2+|A^*|^2\| + \frac{1}{2}\omega(|A^*||A|). \tag{1.4}$$

In [10], Dragomir proved the following numerical radius inequality involving the product of two operators:

$$\omega^r(B^*C) \le \frac{1}{2}\||B|^{2r}+|C^*|^{2r}\|, \qquad r \ge 1. \tag{1.5}$$

For more refined and generalized numerical radius inequalities, the reader may see [2,5, 11,13–15,20,21].

For off-diagonal 2×2 operator matrix $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, Bhunia and Paul [6] proved the following inequalities:

$$\omega\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le \frac{1}{2}\max\{\|B\|,\|C\|\} + \frac{1}{2}\max\{r^{\frac{1}{2}}(|B||C^*|), r^{\frac{1}{2}}(|B^*||C|)\}, \tag{1.6}$$

$$\begin{aligned} \omega^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le &\frac{1}{4}\max\{\||C|^2+|B^*|^2\|, \||B|^2+|C^*|^2\|\} \\ &+ \frac{1}{2}\max\{\omega(|B^*||C|), \omega(|C^*||B|)\}. \end{aligned} \tag{1.7}$$

In this paper, we mainly use the extension Buzano inequality and the generalized Young inequality to give some generalized upper bounds for the numerical radius of off-diagonal 2×2 operator matrices. And our bounds refine inequalities (1.6) and (1.7). Moreover, the conclusion is applied to the non-monic operator polynomials and gives a new bound for the eigenvalues of these operator polynomials.

2 Improved numerical radius inequalities for the off-diagonal 2×2 operators matrices

To prove our main results, we need the following well-known lemmas. The following Lemma is the extension of Buzano inequality and it is introduced in [8].