J. Math. Study
doi: 10.4208/jms.v58n3.25.06

Vol. **58**, No. 3, pp. 338-361
September 2025

A Regularity Result for the Incompressible Elastodynamics with a Free Interface

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Received February 16, 2024; Accepted September 4, 2024;
Published online September 15, 2025.

Abstract. We consider the incompressible inviscid elastodynamics equations with a free surface and establish the regularity of solutions for elastic system. Compared with the previous result on this free boundary problem [Gu X and Wang F, Well-posedness of the free boundary problem in incompressible elastodynamics under the mixed type stability condition, J. Math. Anal. Appl., 2020, 482(1): 123529] in space H^3, we are able to establish the regularity in space $H^{2.5+\delta}$. It is achieved by reformulating the system into the Lagrangian formulation, presenting the uniform estimates for the pressure, the tangential estimates for the system, as well as the divergence and curl estimates.

AMS subject classifications: 76N10, 35Q35, 35D35, 76E19

Key words: Free boundary, incompressible elastodynamics equations, a priori estimates.

1 Introduction

1.1 Eulerian formulation

We consider the equations of ideal incompressible elastodynamics in a moving domain $\Omega(t)\times[0,T]$:

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla p=\mathrm{div}(GG^T) & \text{in } \Omega(t),\\ \partial_t G+u\cdot\nabla G=\nabla u G & \text{in } \Omega(t),\\ \mathrm{div}\ u=0 & \text{in } \Omega(t),\end{cases} \tag{1.1}$$

where $u\in\mathbb{R}^3$ is the velocity field, $G\in\mathbb{M}^{3\times 3}$ is the deformation tensor, G^T denotes the transpose of the matrix G, and p is the fluid pressure of the elastic medium. GG^T is the Cauchy-Green stress tensor. We impose the divergence constraints on the deformation tensor

$$\mathrm{div}\ G^T=0 \qquad \text{in } \Omega(t), \tag{1.2}$$

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this property holds at any time throughout the flow if it is satisfied initially.

It is assumed that $\Omega(t)$ is initially the channel

$$\Omega(0)=\Omega=\mathbb{T}^2\times[0,\tilde{\varepsilon}], \tag{1.3}$$

with the rigid bottom boundary

$$\Gamma_0=\mathbb{T}^2\times\{0\}, \tag{1.4}$$

while the top moving interface $\Gamma_1(t)$ evolves and is initially equal to

$$\Gamma_1=\mathbb{T}^2\times\{\tilde{\varepsilon}\}, \tag{1.5}$$

where $\mathbb{T}^2$ denotes the 2-torus, which can be thought of as the unit square $(0,1)^2$ with periodic boundary conditions.

System (1.1) is supplemented with the following boundary conditions on the free surface $\Gamma_1(t)$:

$$u(\Gamma_1(t))=u\cdot n_1,\quad p=0,\quad G^T\cdot n_1=0,\qquad \text{on } \Gamma_1(t), \tag{1.6}$$

and on the rigid bottom boundary Γ_0:

$$u\cdot n_0=0,\quad G^T\cdot n_0=0,\qquad \text{on } \Gamma_0. \tag{1.7}$$

The first condition of (1.6) is called the kinematic boundary condition which states that the free surface $\Gamma_1(t)$ moves with the velocity of the fluid, where $u(\Gamma_1(t))$ denotes the normal velocity of $\Gamma_1(t)$ and n_1 denotes the outward unit normal vector of $\Gamma_1(t)$. The second and third conditions of (1.6) means that outside the fluid region is the vacuum and the deformation tensor is tangential on the boundary, n_0 denotes the outward unit normal vector of Γ_0.

To establish a priori bounds for the elastodynamics equations, we must impose the Rayleigh-Taylor(R-T) sign condition on the free boundary $\Gamma_1(t)$, namely,

$$-\frac{\partial p}{\partial n_1}\geq\varepsilon_0>0,\qquad \text{on } \Gamma_1(t), \tag{1.8}$$

where ε_0 is a constant. This condition was initially applied by Hao and Wang [10] while proving the a priori energy estimates for the free boundary incompressible elastodynamics equations with H^4 initial data.

Finally, to complete the problem, we impose the initial condition

$$(u,G)=(u_0,G_0)\qquad \text{on } \Omega(0),\qquad \text{and}\qquad \Omega(0)=\Omega. \tag{1.9}$$

The aim of this paper is to establish a priori estimates for solutions with minimal regularity assumptions on the initial data in Lagrangian coordinates. Our analysis in this paper relies on the reformulation of the problem under consideration in Lagrangian coordinates. The benefit of this reformation is to get a key conserved quantity: $\partial_t(F\mathcal{A}^T)=0$. This conserved quantity further enable us to derive two important identities: $F\mathcal{A}^T=G^0$,