J. Math. Study
doi: 10.4208/jms.v58n3.25.07

Vol. **58**, No. 3, pp. 362-379
September 2025

On a Class of Degenerate Nonlinear Elliptic Equations in Weighted Sobolev Spaces

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Received February 16, 2024; Accepted September 4, 2024;
Published online September 15, 2025.

Abstract. In this paper we are interested in the existence of solutions for Dirichlet problem associated to the degenerate nonlinear elliptic equations

$$-\sum_{j=1}^{n} D_j\big[\omega(x)\mathcal{B}_j(x,u,\nabla u)\big]-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x))+b(x,u,\nabla u)\omega(x)+g(x)u(x)$$
$$=f_0(x)-\sum_{j=1}^{n} D_jf_j(x), \qquad \text{in } \Omega$$

in the setting of the weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega)$.

AMS subject classifications: 35J70, 35J60

Key words: Degenerate nonlinear elliptic equations, weighted Sobolev spaces, A_p-weights.

1 Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega)$ (see Definition 2.2) for the Dirichlet problem

$$(P)\begin{cases} Lu(x)=f_0(x)-\displaystyle\sum_{j=1}^{n} D_jf_j(x), & \text{in } \Omega \\ u(x)=0, & \text{on } \partial\Omega \end{cases}$$

where L is the partial differential operator

$$Lu(x)=-\sum_{j=1}^{n} D_j\big[\omega(x)\mathcal{B}_j(x,u(x),\nabla u(x))\big]-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x))$$

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$$+b(x,u,\nabla u)\omega(x)+g(x)u(x),$$

where $D_j=\partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω is a weight function and the functions $\mathcal{B}_j:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j=1,\cdots,n)$, $g:\Omega\to\mathbb{R}$, $b:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ and $a_{ij}:\Omega\to\mathbb{R}$ satisfy the following conditions:

(H1) $x\mapsto\mathcal{B}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$,
$(\eta,\xi)\mapsto\mathcal{B}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H2) there exists a constant $\theta_1>0$ such that $[\mathcal{B}(x,\eta,\xi)-\mathcal{B}(x,\eta',\xi')].(\xi-\xi')\geq\theta_1|\xi-\xi'|^p$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{B}(x,\eta,\xi)=(\mathcal{B}_1(x,\eta,\xi),...,\mathcal{B}_n(x,\eta,\xi))$ (where a dot denotes here the Euclidean scalar product in $\mathbb{R}^n$) and $p>2$.

(H3) $\mathcal{B}(x,\eta,\xi).\xi\geq\lambda_1|\xi|^p$, where $\lambda_1>0$.

(H4) $|\mathcal{B}(x,\eta,\xi)|\leq K_1(x)+h_1(x)|\eta|^{p-1}+h_2(x)|\xi|^{p-1}$, where K_1,h_1 and h_2 are non-negative functions, with h_1 and $h_2\in L^\infty(\Omega)$, and $K_1\in L^{p'}(\Omega,\omega)$ $(1/p+1/p'=1)$.

(H5) $x\mapsto b(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$,
$(\eta,\xi)\mapsto b(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H6) there exists a constant $\theta_2>0$ such that $[b(x,\eta,\xi)-b(x,\eta',\xi')](\eta-\eta')\geq\theta_2|\eta-\eta'|^p$, whenever $\eta,\eta'\in\mathbb{R}$, $\eta\neq\eta'$.

(H7) $b(x,\eta,\xi)\eta\geq\Lambda_1|\eta|^p$, where $\Lambda_1>0$.

(H8) $|b(x,\eta,\xi)|\leq K_2(x)+h_3(x)|\eta|^{p-1}+h_4(x)|\xi|^{p-1}$, where K_2,h_3 and h_4 are non-negative functions, with $K_2\in L^{p'}(\Omega,\omega)$, h_3 and $h_4\in L^\infty(\Omega)$.

(H9) $g/v\in L^\infty(\Omega)$ and $g(x)\geq0$ a.e. $x\in\Omega$, where v is a weight function.

(H10) $a_{ij}:\Omega\to\mathbb{R}$ are measurable, the coefficient matrix $\mathcal{A}(x)=(a_{ij}(x))$ is symmetric and satisfies the degenerate ellipticity condition

$$\lambda_2|\xi|^2v(x)\leq\sum_{i,j=1}^n a_{i,j}(x)\xi_j\xi_j\leq\Lambda_2|\xi|^2v(x) \tag{1.1}$$

for all $\xi\in\mathbb{R}^n$ and almost every $x\in\Omega$, $\lambda_2>0$ and $\Lambda_2>0$ are constants.

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable nonnegative locally integrable functions $\omega=\omega(x)$, $x\in\Omega$ a.e. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E)=\int_E\omega(x)dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces ([2], [3], [4], [6] and [7]).

A class of weights, which is particulary well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt ([11]). These classes have found many usefull applications in harmonic analysis ([12]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p ([10]). There are, in fact, many interesting examples of weights ([9] for p-admissible weights).