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Some p-Adic Hardy Operators and Their Commutators

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Abstract. In this paper, we will study the sharp estimates of p-adic weighted Hardy operator on central and noncentral weighted p-adic Morrey spaces. Moreover, we can obtain the sharp bound of generalized p-adic Hardy operator. In addition, the commutator that is generated by the generalized p-adic Hardy operator and the central BMO function is also bounded on p-adic Morrey spaces.

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1 Introduction

Since Hardy operator had some applications in many fields of mathematics, it has become an important research direction of harmonic analysis. Let f be a nonnegative integrable function on $\mathbb{R}^+$, the classical Hardy operators are defined by

$$Hf(x):=\frac{1}{x}\int_0^x f(t)dt \qquad \text{and} \qquad H^*f(x):=\int_x^\infty \frac{f(t)}{t}dt, \qquad \text{where } x>0.$$

Moreover, these two operators are conjugate operators, which means that

$$\int_0^\infty (Hf)(x)g(x)dx=\int_0^\infty f(x)(H^*g)(x)dx,$$

where $f\in L^p(\mathbb{R}^+),g\in L^q(\mathbb{R}^+),1<p<\infty$ and $1/p+1/q=1$.

To estimate the Hardy operators, G. H. Hardy established the Hardy's integral inequalities in 1920, which can be stated as follows.

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Theorem 1.1 ([12]). *Let $1<p<\infty$ and q be the conjugate exponent of p. Then we have*

$$\|Hf\|_{L^p} \le \frac{p}{p-1}\|f\|_{L^p} \qquad \text{and} \qquad \|H^*f\|_{L^q} \le \frac{p}{p-1}\|f\|_{L^q}.$$

What's more,

$$\|H\|_{L^p\to L^p} = \|H^*\|_{L^q\to L^q} = \frac{p}{p-1}.$$

For the past few years, Hardy's integral inequalities have attracted considerable attention from scholars. Furthermore, a number of papers have appeared on the alternative proofs and applications of Hardy's inequalities. As for the earlier development of Hardy's inequalities and their important applications in analysis, please see [13].

After the Hardy operator was defined by Hardy, Carton-Lebrun and Fosset [4] first proposed the weighted Hardy operator in 1984. Let $\varphi:[0,1]\to[0,\infty)$, the weighted Hardy operator is defined as

$$U_\varphi f(x) := \int_0^1 f(tx)\varphi(t)dt, \qquad \text{where } x\in\mathbb{R}^n,$$

and the dual operator of U_φ is V_φ, which is defined by

$$V_\varphi f(x) := \int_0^1 f(x/t)t^{-n}\varphi(t)dt, \qquad \text{where } x\in\mathbb{R}^n.$$

The weighted Hardy operator is closely related to the classical Hardy-Littlewood maximal operator, so it had drawn some attention in recent years. Carton-Lebrun and Fosset [4] proved that U_φ is bounded on $\mathrm{BMO}(\mathbb{R}^n)$ in 1984. Moreover, Xiao [43] obtained the following theorem.

Theorem 1.2 ([43]). *Let $1<p<\infty$ and U_φ be bounded in $L^p(\mathbb{R}^n)$ if and only if*

$$\int_0^1 t^{-\frac{n}{p}}\varphi(t)dt<\infty.$$

After this, Wu [42] studied the weighted Hardy operator on generalized Morrey space in 2011, and she proved that the weighted Hardy operator is bounded on this space as stated below.

Definition 1.1 ([42]). *Let $1\le p<\infty$ and $\omega(r)\in\mathbb{R}^n$ be a nonnegative integrable function satisfying $\int_0^x \frac{\omega(t)}{t}dt \le c\omega(x)$, and $\frac{\omega(r)}{r^n}$ is decreasing. Then the generalized Morrey space $L^{p,\omega}(\mathbb{R}^n)$ is defined as*

$$L^{p,\omega}(\mathbb{R}^n) := \{f\in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,\omega}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{L^{p,\omega}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0}\frac{1}{\omega^{1/p}(r)}\|f\|_{L^{p,\omega}(B(x,r))}.$$