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A Note on the Global Existence for the Strong Solution of the Stochastic Three-Dimensional Axially Symmetric Euler Equations

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Abstract. This paper establishes the global existence of the pathwise solutions for the stochastic 3-D incompressible Euler equations with the axially symmetric initial data and axially symmetric additive noise.

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Key words: Stochastic Euler equation, axially symmetric, pathwise solution, global existence.

1 Introduction

The 3-D Euler equations are the classical model for the motion of an inviscid, incompressible, homogeneous fluid, which read:

$$\begin{cases} \partial_t \mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \nabla\cdot\mathbf{u}=0, \\ \mathbf{u}(0,x)=\mathbf{u}_0(x). \end{cases} \tag{1.1}$$

where $\mathbf{u}$ denotes the velocity field, and p the pressure scalar field. The global existence and uniqueness of smooth solution is still open for the Euler equations, see, for instance [2,3,5,7] and references therein. Some positive results are available for the 3D flows with certain geometry constraints as the so-called axially symmetric flows.

In cylindrical coordinates, the velocity $\mathbf{u}$ is written as:

$$\mathbf{u}(t,x)=u^r e_r+u^\theta e_\theta+u^3 e_3, \tag{1.2}$$

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where

$$e_r=\Big(\frac{x_1}{r},\frac{x_2}{r},0\Big),\qquad e_\theta=\Big(-\frac{x_2}{r},\frac{x_1}{r},0\Big),\qquad e_3=(0,0,1),\qquad \text{and}\qquad r=\sqrt{x_1^2+x_2^2}.$$

Axially symmetry without swirl means that $u^\theta=0$ and u^r, u^3 are independent of θ. The global in time existence of axially symmetric solutions to (1.1) is first proven by authors Ukhovski and Yudovich in [16]. Then Shirota and Yanagisawa [15] proved the global existence is H^s $(s>5/2)$, which is the optimal result in Sobolev space. Danchin [8], Abidi, Hmidi and Keraani [1] obtained global existence and uniqueness in Lorentz space and critical Besov space, respectively.

Just as [9] said, in view of the wide usage of stochastic in fluid dynamic, there is an essential need to improve mathematical foundations of the stochastic partial differential equations of fluid flow, and in particular to study inviscid models such as the stochastic Euler equations. The stochastic three-dimensional Euler equations read as follows:

$$\begin{cases} d\mathbf{u}+(\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p)dt=\sum\limits_{i\geq 1}\sigma_i dW_i, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \nabla\cdot\mathbf{u}=0,\\ \mathbf{u}|_{t=0}=\mathbf{u}_0, \end{cases} \tag{1.3}$$

where W_i are a collection of one-dimensional independent Brownian motionsn motions over a given sample space Ω. For this system, Kim [12] and Mikulevicius [13] considered the local solutions with additive noise in a weighted Hölder space and $H^s(\mathbb{R}^3)$ $(s>5/2)$, respectively. Glatt-Holtz and Vicol established the local existence of the pathwise solution in a bounded domain and showed the regularizing effect of the linear multiplicative noise in [9] .

The aim of this paper is to establish global existence of pathwise solutions for the stochastic 3-D incompressible Euler equations with the axially symmetric initial data and axially symmetric additive noise.

Organization of the paper: In Section 2, we introduce some mathematics backgrounds and the precise definition of strong solution to (1.3), along with the statement of our main result. In Section 3, we derive the form of the stochastic axially symmetric Euler equations and show the stochastic version of blow-up criteria for stochastic Euler equations which has been proved in [9]. At last, we prove Theorem 2.2 in Section 4.

2 Notation and statement of the main result

We define, for $s\geq 0$,

$$\mathcal{H}^s=\{f\in H^s(\mathbb{R}^3)^3|\nabla\cdot f=0\},$$

which is a close subspace of H^s. The symbol Π stands for the projection from $H^s(\mathbb{R}^3)^3$ to $\mathcal{H}^s$. Throughout this paper, a stochastic basis $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbf{P})$ is given and fixed, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous filtration over the probability space $\{\Omega,\mathcal{F},\mathbf{P})$ such that $\mathcal{F}_0$