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On the Fundamental Group of Non-Collapsed Ancient Ricci Flows

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Abstract. We show that any manifold admitting a non-collapsed, ancient Ricci flow must have finite fundamental group. This generalizes what was known for κ-solutions in dimensions 2, 3. We furthermore show that this fundamental group must be a quotient of the fundamental group of the regular part of any tangent flow at infinity.

AMS subject classifications: 53E20

Key words: Ricci flow, singularity model, fundamental group, kappa-solution, tangent flow.

1 Introduction

A central goal in the study of Ricci flow is the study of its singularity formation. In dimension 3 this was carried out successfully by Perelman [15] and led to the construction of a Ricci flow with surgery [16]. In higher dimensions, progress towards this goal was recently obtained by the author in [2–4], where singularities were characterized by (possibly singular) gradient shrinking solitons that arise as blow-up models along specific sequences of points and scales. In the same work it was shown that it is—in fact—possible to take blow-up limits along any sequence of points and scales and that this limit is given by a non-collapsed, ancient Ricci flow with a possible singular set of codimension ≥ 4. In order to obtain a characterization of the part of the manifold that becomes singular, it becomes necessary to study these non-collapsed, ancient flows in more detail. A better understanding of these flows may allow us in the future to perform a successful surgery construction and possibly derive useful topological consequences on the underlying manifold.

This short paper is the beginning of study of non-collapsed, ancient Ricci flows. In dimension 3, such flows are *κ-solutions*, which arose in Perelman's work [16] and are now

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fully classified [5, 6]. In higher dimensions, however, very little has been known about such flows.

In this paper we will focus, for simplicity, on non-collapsed ancient flows that are *non-singular* and have *complete time-slices* and *bounded curvature on compact time-intervals.* This regularity assumption was considered frequently in dimension 3, for example in the study of κ-solutions where it turned out to be natural. We also remark that in forthcoming work, we will establish a more general theory of *non-compact, singular* flows and our techniques will readily generalize to such flows.

Let us now summarize our results. We will show that the underlying manifold of any non-collapsed, ancient flow with complete time-slices and bounded curvature on compact time-intervals has finite fundamental group. This generalizes what has been known to be true for κ-solutions in dimension 2, 3. We will also show that this fundamental group must be the quotient of the fundamental group of the regular part of any tangent flow at infinity, where the latter is given by a singular gradient shrinking soliton. This implies further restrictions on the fundamental group of the flow in certain cases. Lastly, we show that any tangent flow at infinity is a quotient of the corresponding tangent flow at infinity of the universal covering flow and derive an identity on the order of the fundamental group of the original flow in terms of the Nash entropies at infinity of both flows.

Recall that an ancient Ricci flow $(M,(g_t)_{t\le 0})$ with complete time-slices and bounded curvature on compact time-intervals is called *non-collapsed* if the following equivalent conditions are satisfied [11]:

1. We have the following entropy bound for some uniform Y
$$\mu(g_t,\tau)\ge -Y, \qquad \text{for all } \ t\le 0, \tau>0.$$
2. We have the entropy bound
$$\liminf_{\tau\to\infty}\mu(g_{-\tau},\tau)>-\infty.$$
3. For some (or any) $(x,t)\in M\times\mathbb{R}_{\le 0}$ we have the following bound on the pointed Nash entropy
$$\liminf_{\tau\to\infty}\mathcal{N}_{x,t}(\tau)>-\infty.$$

Moreover, $(M,(g_t)_{t\le 0})$ is automatically non-collapsed if it arises as the blow-up limit of a Ricci flow $\left(M',(g'_t)_{t\in[0,T)}\right)$ on a compact manifold M'. Any such non-collapsed, ancient flow has a tangent cone at infinity, which is given by the flow of a singular gradient shrinking soliton. Such a soliton is described by a *singular space,* i.e., a tuple $(X,d,\mathcal{R}_X,g_X,f)$, where (X,d) is a metric length space, $\mathcal{R}_X\subset X$ is a maximal open subset equipped with a smooth manifold structure and Riemannian metric g_X such that (X,d)