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A Blow up Solution of the Navier-Stokes Equations with a Super Critical Forcing Term

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Dedicated to Professor Gang Tian on occasion of his 65th birthday.

Abstract. A forced solution v of the axially symmetric Navier-Stokes equation in a finite cylinder D with suitable boundary condition is constructed. The forcing term, whose order of scaling is slightly worse than the critical order -2, is in the mildly super critical space $L^q_t L^1_x$ for all $q>1$. The velocity, which is smooth until its final blow up moment, is in the energy space throughout.

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Key words: Navier-Stokes equations, blow up example with supercritical forcing term.

1 Statement of result and proof

The Navier-Stokes equation (NS) describing the motion of viscous incompressible fluids in a domain $D\subset\mathbb{R}^3$ can be written as

$$\mu\Delta v - v\nabla v - \nabla P - \partial_t v = f, \quad \mathrm{div}\, v = 0, \qquad \text{in } D\times(0,\infty),\ v(\cdot,0)=v_0(\cdot). \tag{1.1}$$

The unknowns are the velocity v and the pressure P. $\mu>0$ is the viscosity constant, which will be taken as 1; f is a given forcing term. In order to solve the equation, suitable boundary conditions should be given if $\partial D\neq\emptyset$.

Thanks to Leray [12], if $D=\mathbb{R}^3$, $v_0\in L^2(\mathbb{R}^3)$, $f=0$, the Cauchy problem has a solution in the energy space $(v,\nabla v)\in(L^\infty_t L^2_x, L^2_{xt})$. However, in general it is not known if such solutions stay bounded or regular for all $t>0$ for regular initial values. Neither the uniqueness of such solutions is known so far. A typical existence theorem for regular solutions always involves a small parameter in the initial condition, as a perturbation of a known regular solution.

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Given a semi-open time interval $[0,T)$, in this short note, we construct a regular solution of (1.1) in a finite cylinder with a forcing term in the super critical space $L^q_t L^1_x$ for all $q>1$. The velocity is in the energy space at the final time T when it blows up. Solutions beyond the energy space or super critical forcing terms have been investigated before. For a forcing term in the supercritical space $L^1_t L^2_x$, in the paper [1], non-uniqueness of (1.1) in the energy space is established. Since $2<\frac{3}{1}+\frac{2}{q}<\frac{3}{2}+\frac{2}{1}$ for large q, the forcing term here is actually milder than that in [1] with respect to the standard scaling. Let us recall that a forcing term in local $L^q_t L^p_x$ space is super critical, critical and sub critical if $\frac{3}{p}+\frac{2}{q}>,=,<2$ respectively. Essentially speaking, under the standard scaling, a forcing term is super critical, critical and sub critical if its scaling order is less than, equal to or greater to -2 respectively. In the homogeneous case $f=0$, non-uniqueness was proven for some solutions in the space $L^\infty_t L^2_x$ in [2]. Let us mention that an instantaneous finite time blow up example was constructed in [16] for a special cusp domain. The solution is in the energy space and the forcing term (Ampere force) is actually subcritical. These results indicate that if pertinent function spaces or the domain are sufficiently bad, then something bad can happen to the solutions.

Now let us describe the blow up solution in detail. The solution is axially symmetric, namely v and P are independent of the angle in a cylindrical coordinate system (r,θ,x_3). That is, for $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $r=\sqrt{x_1^2+x_2^2}$, $\theta=\arctan(x_2/x_1)$, and the basis vectors e_r,e_θ,e_3 are:

$$e_r=(x_1/r,x_2/r,0),\qquad e_\theta=(-x_2/r,x_1/r,0),\qquad e_3=(0,0,1).$$

In this case, solutions can be written in the form of

$$v=v_r(r,x_3,t)e_r+v_\theta(r,x_3,t)e_\theta+v_3(r,x_3,t)e_3 \tag{1.2}$$

and the forcing term can be written as

$$f=f_r(r,x_3,t)e_r+f_\theta(r,x_3,t)e_\theta+f_3(r,x_3,t)e_3 \tag{1.3}$$

Therefore, v_r, v_3 and v_θ satisfy the axially symmetric Navier-Stokes equations

$$\begin{cases}\left(\Delta-\dfrac{1}{r^2}\right)v_r-(v_r\partial_r+v_3\partial_{x_3})v_r+\dfrac{(v_\theta)^2}{r}-\partial_rP-\partial_tv_r=f_r,\\ \left(\Delta-\dfrac{1}{r^2}\right)v_\theta-(v_r\partial_r+v_3\partial_{x_3})v_\theta-\dfrac{v_\theta v_r}{r}-\partial_tv_\theta=f_\theta,\\ \Delta v_3-(v_r\partial_r+v_3\partial_{x_3})v_3-\partial_{x_3}P-\partial_tv_3=f_3,\\ \dfrac{1}{r}\partial_r(rv_r)+\partial_{x_3}v_3=0,\end{cases} \tag{1.4}$$

which will be abbreviated as ASNS.

If the swirl $v_\theta=0$ and f is sufficiently regular, then it is well known ([11,22]), that finite energy solutions to the Cauchy problem of (1.4) in $\mathbb{R}^3$ are smooth for all time $t>0$. See also [15]. Some existence results with a small parameter can be found in [7,8] and [24].