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Discrete Four-Order Schrödinger Equation on the Hexagonal Triangulation

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Abstract. In this paper, we establish a decay estimate for the discrete four-order Schrödinger equation on the hexagonal triangulation with $\gamma=0$. The proof is based on the uniform estimates of oscillatory integrals, as developed by Karpushkin, along with a key result by Varchenko. Our result is to show the $l^1\to l^\infty$ dispersive decay rate is $\langle t\rangle^{-\sigma}$ for any $0<\sigma<\frac{1}{2}$. Additionally, we provide estimates for the inhomogeneous discrete fourth-order Schrödinger equation with $\gamma=0$.

AMS subject classifications: 35Q55, 37K60

Key words: Schrödinger equation, graph, Strichartz estimate.

1 Introduction

Karpman [23], Karpman-Shagalov [24] and Fukumoto-Moffatt [12] introduced the fourth-order nonlinear Schrodinger equation when studying the effect of high-order dispersion on the propagation of intense laser in massive media with Kerr nonlinearization and the motion of vortex filament in incompressible fluid. The equation is given as follows

$$i\partial_t u+\Delta^2 u-\gamma\Delta u=F(u,t), \tag{1.1}$$

where $u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$ is a complex-valued function and $\gamma\in\{-1,0,+1\}$. Obviously, it is a natural consideration to generalize the discrete analogs of fourth-order Schrödinger equation.

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Suppose $G=(V,E)$ is a connected undirected graph, where V is the vertex set and E is the edge set of G, we write $i\sim j$ if $\{i,j\}\in E$. The following equation is called discrete fourth-order Schrödinger equation on G:

$$\begin{cases} i\partial_t u_v(t)+\Delta_G^2 u_v(t)-\gamma\Delta_G u_v(t)=0, & \forall v\in V, \\ u(0)=\psi, \end{cases} \tag{1.2}$$

where $\gamma\in\mathbb{R}$ and the discrete Laplacian operator Δ_G is given by

$$\Delta_G f_i=\sum_{j:i\sim j}\omega_{ij}\left(f_j-f_i\right), \tag{1.3}$$

where ω_{ij} is the weight on E.

The discrete Laplace operator Δ_G occurs in a wide variety of physical environments, such as loop quantum gravity [7], Ising model [22] and discrete dynamical systems [11, 38]. Besides, it also plays a pivotal role in computer science, with applications in image processing, clustering, and semi-supervised learning [31, 33]. Furthermore, the discrete Schrödinger equation and its nonlinear variant are also applied in problems such as Davydov's soliton in biophysics, nonlinear optical couplers, and waveguide arrays. In the 2000s, these equations found renewed significance in an entirely different setting within Bose-Einstein condensates (BECs) in optical lattices, representing some of the most exciting nonlinear phenomena in this new state of matter. This development was particularly noteworthy, as the realization of BECs was awarded the Nobel Prize in Physics in 2001, with another following in 2003 for their connection to superfluidity [29].

In this paper, for simplicity of writing, we will write Δ_G also as Δ. Moreover, we consider the infinite graph G with a physical background in Figure 1, which is a regular graph with degree 6 on plane and the physical interpretation of graph G is that an atom is influenced by its nearest atoms. Besides, the fourth-order Schrödinger equation in the form of Equation. (1.2) with the graph G in Fig. 1 serves as an important model in physics to describe the vibrations of atoms within crystals [11, 38].

Finding Strichartz estimates for linear dispersive equations is a classical problem. Strichartz estimates are a family of inequalities that establish the size and decay of solutions in mixed norm Lebesgue spaces for linear dispersive partial differential equations. They were first noted by Robert Strichartz [36] and arose from connections to the Fourier restriction problem. A typical Strichartz estimate is an $l^1\to l^\infty$ estimate of the form

$$\|u(t)\|_{l^\infty}\leq C\langle t\rangle^{-\sigma}\|\psi\|_{l^1}, \tag{1.4}$$

where $\langle t\rangle=(1+|t|)$ and C is a constant.

In Euclidean space $\mathbb{R}^d$, it is well-known that the $L^1\to L^\infty$ estimate of the solution to the Schrödinger equation is $\sigma=\frac{d}{2}$. For the wave equation [19], $\sigma=\frac{d-1}{2}$. For the biharmonic Schrödinger operator [2], $\sigma=\frac{d}{4}$. However, the analysis on $\mathbb{Z}^d$ which is the discrete analog to Euclidean space $\mathbb{R}^d$ is extremely dependent on converting the phase function of the oscillation integral into a simple polynomial, which is the canonical form for the singularity