J. Math. Study
doi: 10.4208/jms.v58n4.25.04

Vol. **58**, No. 4, pp. 459-478
December 2025

$W^{3,p}$ Estimate of Degenerate Complex Monge-Ampère Equations

Jianchun Chu[1] and Feng Wang [2,*]

[1] *School of Mathematical Sciences, Peking University, Yiheyuan Road 5, Beijing 100871, China;*
[2] *School of Mathematical Sciences, Zhejiang University, Yuhangtang Road 866, Hangzhou 310000, China.*

Received December 23, 2024; Accepted April 29, 2025;
Published online December 15, 2025.

Abstract. Degenerate complex Monge-Ampère equations arise naturally in the study of geometry of singular varieties. In this paper, we prove gradient estimate and $W^{3,p}$ estimate for a class of degenerate complex Monge-Ampère equations.

AMS subject classifications: 32W20, 32Q15

Key words: Complex Monge-Ampère equations, Kähler manifolds.

1 Introduction

Let (Y,ω_Y) be a compact n-dimensional normal Kähler variety. We consider the complex Monge-Ampère equation

$$\begin{cases} (\omega_Y+\sqrt{-1}\partial\bar{\partial}u)^n=e^F\omega_Y^n, \\ \omega_Y+\sqrt{-1}\partial\bar{\partial}u>0, \\ \sup_X u=0, \end{cases} \tag{1.1}$$

where F is function on Y with some certain regularity. To study this equation, it suffices to investigate its pull-back equation on some smooth manifold under resolution of singularities. More precisely, thanks to Hironaka's theorem [8], there exists a resolution of the singularities of Y:

$$\mu:(X,\omega)\to(Y,\omega_Y),$$

*Corresponding author. *Email addresses:* jianchunchu@math.pku.edu.cn (Chu J), wfmath@zju.edu.cn (Wang F)

where (X,ω) is a compact n-dimensional Kähler manifold. Define $\alpha=\mu^*\omega_Y$. It is well-known that $[\alpha]$ is a big and semi-positive class on X, and that

$$E_{nK}(\alpha)=\mu^{-1}(Y_{\text{Sing}}),$$

where $E_{nK}(\alpha)$ denotes the non-Kähler locus of $[\alpha]$ and Y_{Sing} denotes the singular part of variety Y. For convenience, we identify μ^*u and μ^*F with u and F. Then the pull-back equation of (1.1) is

$$\begin{cases} (\alpha+\sqrt{-1}\partial\bar{\partial}u)^n=e^F\alpha^n, \\ \alpha+\sqrt{-1}\partial\bar{\partial}u>0, \\ \sup_X u=0. \end{cases} \tag{1.2}$$

However, this equation is degenerate along $E_{nK}(\alpha)$. So we need to consider its perturbation. For $\varepsilon>0$, the perturbed equation is

$$\begin{cases} (\alpha+\varepsilon\omega+\sqrt{-1}\partial\bar{\partial}u)^n=e^{F_\epsilon}(\alpha+\varepsilon\omega)^n, \\ \alpha+\varepsilon\omega+\sqrt{-1}\partial\bar{\partial}u>0, \\ \sup_X u=0, \end{cases} \tag{1.3}$$

where $F_\epsilon=F+c_\epsilon$ for some small constant c_ϵ. We will simply denote F_ϵ by F. When α is a Kähler metric, the non-Kähler locus $E_{nK}(\alpha)$ is empty, so there is no need to consider perturbation (1.3). In this case, Y_{Sing} is empty, i.e., (Y,ω_Y) is actually a compact Kähler manifold. In [4], Chen-He established W^{3,p_0} estimate for the solution of (1.2) when the RHS $F\in W^{1,p_0}(Y,\omega_Y)$ where $p_0>2n$. Later, Chen-Cheng [3] proved the gradient estimate under weaker assumptions of F. They established L^p estimate of the gradient when $F\in C^0(Y,\omega_Y)$, and L^∞ estimate of gradient when $F\in C^0(Y,\omega_Y)$ satisfies $\int_0^1\frac{\omega_F^2(r)}{r}dr<\infty$, where $\omega_F(r)$ denotes the modulus of continuity of F with respect to ω_Y.

It is very interesting to generalize the above works to more general setting that Y_{Sing} is not empty. This means $E_{nK}(\alpha)=\mu^{-1}(Y_{\text{Sing}})$ is not empty, and α is never Kähler. As mentioned above, (1.2) is degenerate along $E_{nK}(\alpha)$, so we consider (1.3) and hope to establish some a priori estimates for the solution u away from $E_{nK}(\alpha)$, which is independent of ε.

In Section 2, we will construct a cut-off function ψ of $E_{nK}(\alpha)$ on X such that (see Lemma 2.1)

$$E_{nK}(\alpha)=\{\psi=-\infty\}.$$

We will prove the following estimates:

Theorem 1.1. *Let u be a smooth solution of* (1.3) *and $\omega_F(r)$ be the the modulus of continuity of F with respect to ω.*

(i) For any $p>0$, there exist constants Λ and C_p depending only on $\|F\|_{C^0(X,\omega)}$, $\omega_F(r)$, p, α and (X,ω) such that

$$\int_X e^{\Lambda\psi}|\nabla u|^p\omega^n\le C_p.$$

Here Λ does not depend on p.