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p-Laplace Equations, p-Superharmonic Functions, and Applications in Conformal Geometry

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Abstract. In this paper, we give an exposition of our recent work on nonlinear potential theory in conformal geometry. We apply nonlinear potential theory to study p-Laplace equations arising from conformal geometry and, in particular, the problems related to the asymptotic behavior near and the size of singularities in conformal geometry.

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1 Introduction of p-Laplace equations in conformal geometry

In conformal geometry one often encounters the Schouten curvature tensor on a manifold (M^n,g)

$$A=\frac{1}{n-2}\left(Ric-\frac{1}{2(n-1)}Rg\right), \tag{1.1}$$

where Ric stands for the Ricci curvature tensor and $R=\mathrm{Tr}_g Ric$ is the scalar curvature. For a good reason, one uses notation $J=\frac{R}{2(n-1)}$. Here, we want to call the attention to the intermediate Schouten curvature tensor ([19,20])

$$A^{(p)}=(p-2)A+Jg \tag{1.2}$$

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for $p\in(1,\infty)$. For $\bar{g}=u^{\frac{4(p-1)}{n-p}}g$ and $p\neq n$,

$$\begin{aligned}A^{(p)}[\bar{g}]=&A^{(p)}-\frac{2(p-1)}{n-p}\left[\frac{\Delta u}{u}g+(p-2)\frac{D^2u}{u}\right]\\&+\frac{2(p-1)}{n-p}\left[\left(1-(n+p-4)\frac{p-1}{n-p}\right)\frac{|\nabla u|^2}{u^2}g\right.\\&\left.+(p-2)\left(1+\frac{2(p-1)}{n-p}\right)\frac{\nabla u\otimes\nabla u}{u^2}\right].\end{aligned}\tag{1.3}$$

Multiplying $u|\nabla u|^{p-2}\frac{u_i}{|\nabla u|}\frac{u_j}{|\nabla u|}$ to and summing up on both sides, we arrive at the p-Laplace equations in conformal geometry

$$-\Delta_p u+\frac{n-p}{2(p-1)}S^{(p)}(\nabla u)u=\frac{n-p}{2(p-1)}(S^{(p)}(\nabla u))[\bar{g}]u^q,\tag{1.4}$$

where

$$S^{(p)}(\nabla u)=|\nabla u|^{p-2}A^{(p)}(\nabla u),\qquad q=\frac{2p(p-1)}{n-p}+1,$$

and $A^{(p)}(\nabla u)$ is the $A^{(p)}$ curvature in the direction ∇u. When $p=n$, we realize $A^{(n)}=Ric$, and the n-Laplace equation is

$$-\Delta_n\phi+|\nabla\phi|^{n-2}Ric(\nabla\phi)=(|\nabla\phi|^{n-2}Ric(\nabla\phi))[\bar{g}]e^{n\phi},\tag{1.5}$$

where $\bar{g}=e^{2\phi}g$. Recall

$$\Delta_p u=\mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

For $p=n=2$, (1.5) goes back to the Gauss curvature equation

$$-\Delta\phi+K=K[\bar{g}]e^{2\phi},$$

where $\bar{g}=e^{2\phi}g$. For $p=2$ and $n\geq 3$, the intermediate Schouten curvature goes back to the scalar curvature and the p-Laplace equation (1.4) goes back to the scalar curvature equation

$$-\Delta u+\frac{n-2}{4(n-1)}Ru=\frac{n-2}{4(n-1)}R[\bar{g}]u^{\frac{n+2}{n-2}},$$

where $\bar{g}=u^{\frac{4}{n-2}}g$. For $p>n$, the p-Laplace equation for the intermediate Schouten curvature is still valid

$$-\Delta_p u+\frac{n-p}{2(p-1)}S^{(p)}(\nabla u)u=\frac{n-p}{2(p-1)}(S^{(p)}(\nabla u))[\bar{g}]u^q$$

for $\bar{g}=u^{-\frac{4(p-1)}{p-n}}g$ and $q=-\frac{2p(p-1)}{p-n}+1<0$. And, when taking $p\to\infty$, we arrive at the infinite Laplace equation on Schouten curvature A

$$-\Delta_\infty u-\frac{1}{2}|\nabla u|^2A(\nabla u)u=-\frac{1}{2}(|\nabla u|^2A(\nabla u))[\bar{g}]u^{-7}$$

for $\bar{g}=u^{-4}g$. Recall $\Delta_\infty u=u_{ij}u_iu_j$.