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Fukaya-Yamaguchi Conjecture in Dimension Four

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Abstract. Fukaya and Yamaguchi conjectured that if M^n is a manifold with nonnegative sectional curvature, then the fundamental group is uniformly virtually abelian. In this short note we observe that the conjecture holds in dimensions up to four.

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1 Statement of the main result

Theorem 1.1. *Let M^n be a smooth manifold with nonnegative sectional curvature and $n \leq 4$. Then there exists an abelian subgroup $A \leq \pi_1(M)$ of the fundamental group with universally bounded index $[\pi_1(M):A] \leq C(n)$.*

It is interesting to note that the above fails if one only assumes $\mathrm{Ric} \geq 0$, see [12] and more recently [3] for examples in the closed case. In the case of $\mathrm{Ric} \geq 0$ the fundamental group may not even be finitely generated, see for instance [1,2]. However, it is unknown if the infinitely generated component must be abelian; for instance, it is unknown if there exists a normal abelian subgroup $A \leq \pi_1(M)$ such that the quotient $\pi_1(M)/A$ is finitely generated.

2 Proof of the main result

By the Cheeger-Gromoll soul Theorem [5], M^n deformation retracts onto a compact totally geodesic submanifold. In particular, it is homotopically equivalent to a compact

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manifold with nonnegative sectional curvature of dimension $\leq n$. Hence we can assume without loss of generality that M^n is compact.

We break the proof down into two basic cases, which is whether or not the universal cover $\tilde{M}^n$ is compact or noncompact. Let us first deal with the case when $\tilde{M}^n$ is compact, where in fact we will prove a slightly more general result about effective actions. This generalization will prove useful in the noncompact context:

Lemma 2.1. *Let $(\tilde{M}^n, g)$ be a simply connected manifold with $n \leq 4$, $\sec \geq -1$ and $\operatorname{diam} \leq D$. Then any finite group Γ acting smoothly and effectively on $\tilde{M}$ admits an abelian subgroup $A \leq \Gamma$ which is generated by at most $C(n,D)$ elements and whose index is uniformly bounded $[\Gamma : A] \leq C(n,D)$.*

Remark 2.1. Note that if $\sec \geq 0$ and $\tilde{M}^n$ is compact, then we may rescale in order to assume $\operatorname{diam}(\tilde{M}^n) \leq 1$. In particular, we have that A is generated by at most $C(n)$ generators with $[\Gamma : A] \leq C(n)$.

Proof of Lemma 2.1. Let us first observe that in the case $n=2$ we have that $M^2 = S^2$ and in the case $n=3$ we have that $M^3 = S^3$ as they are simply connected closed manifolds†. In the case $n=4$, let us recall that the Euler characteristic of a simply connected four manifold M^4 satisfies

$$\chi\left(M^4\right) = 2 + b_2 \geq 2 > 0. \tag{2.1}$$

In particular, a simply connected four manifold has positive Euler characteristic. Let us now appeal to the results of Mundet i Riera [10]. In the cases where M is either an integral homotopy sphere or has nonzero Euler characteristic, we have that $\operatorname{Diff}(M)$ is a Jordan space. More precisely and effectively, by [10, Theorem 1.2] we have that if Γ is any smooth effective action on M then there exists an abelian group $A \leq \Gamma$ and C depending only on the dimensions of M and $H^*(M,\mathbb{Z})$ such that:

(i) A is generated by at most C elements;

(ii) $[\Gamma : A] \leq C$.

If we combine with Gromov's betti number estimates [8, Theorem 0.2B], which bounds for us the dimension of $H^*(M,\mathbb{Z})$, this finishes the proof of Lemma 2.1.

Let us make the observation that in the case that Γ is an oriented and free action on M^4, the result is even easier as one gets directly the order bound on Γ:

$$2|\Gamma| \leq \chi\left(M^4/\Gamma\right)|\Gamma| = \chi\left(M^4\right) = 2 + b_2. \tag{2.2}$$

In the case $n=2$ or $n=3$ we may also have instead appealed to [6, Theorem E] in order to make the required conclusions. □

†It will be enough that that M^3 is an integral homology sphere, so we do not really need to appeal to Perelman's proof of the Poincare conjecture.