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Limit of the Kähler-Ricci Flow

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Abstract. One of the significant motivations for studying the Kähler-Ricci flow is its relation to the Analytic Minimal Model Program as initiated by Gang Tian. Carrying out this classification program requires careful analysis of the flow metric, particularly when it encounters singularities. In this note, we survey some results pertaining to the limit for the Kähler-Ricci flow.

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1 Introduction

Let (X,ω_0) be a Kähler manifold and $\omega(t)$ a one-parameter family of Kähler forms satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega)+\nu\omega; \\ \omega(0)=\omega_0, \end{cases} \tag{1.1}$$

where the variable ν is chosen to be $0,-1$ or 1. When $\nu=0$, it's called unnormalised Kähler-Ricci flow, parallel to the unnormalised Ricci flow. When nonzero, it's then a flow normalised for convenience in the studies on corresponding settings.

Remark 1.1. Following the standard convention in this field, we will interchangeably refer to the metric g and its associated Kähler form ω.

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One of the motivators for studying the Kähler-Ricci flow is the Analytic Minimal Model Program initiated by Tian in [14] with the more detailed proposal by Song-Tian [12]. In this program, one aims to classify compact complex varieties up to birational equivalence by a best representative called a minimal model.

In addition to topological classification, the Kähler-Ricci flow aims to construct canonical metrics on these models. From observing (1.1), stationary solutions are Einstein and Calabi-Yau metrics. Understanding when manifolds can admit such metrics is a central aim in differential geometry and is entwined with a rich history. A notable example is the resolution of the Calabi conjecture in the Calabi-Yau Theorem [23].

In this note, we survey some results pertaining to the limit for the Kähler-Ricci flow. The main aim of these notes is to highlight conditions under which the flow converges to a canonical limit, such as a Kähler-Einstein or Calabi-Yau metric. In such cases, the limiting behaviour is independent of the initial metric. For cases where the existence of a canonical limiting metric remains a problem, we can still look towards understanding the uniqueness of the limit.

2 Background

We begin by defining

$$H^{1,1}(X;\mathbb{R})=\frac{\{\text{ closed real }(1,1)-\text{forms on }X\}}{\sqrt{-1}\partial\bar{\partial}C^{\infty}(X,\mathbb{R})}. \tag{2.1}$$

We can identify this cohomology space as a subspace of the $H^2(X;\mathbb{R})$ de Rham cohomology by the Hodge theory and $\partial\bar{\partial}$-lemma. Within it the Kähler cone defines a subset

$$\mathcal{K}:=\left\{[\omega]\,|\,\omega+\sqrt{-1}\partial\bar{\partial}\varphi>0\right\} \tag{2.2}$$

containing all classes with a positive representative. Any Kähler metric determines a class in the Kähler cone. Furthermore, the Ricci form is defined by

$$\mathrm{Ric}(\omega)=\sqrt{-1}\sum_{i,i=1}^{n}R_{i\bar{j}}dz_i\wedge d\bar{z}_j, \tag{2.3}$$

where $R_{i\bar{j}}$ is a component of the Ricci tensor, locally given by

$$R_{i\bar{j}}=-\partial_i\partial_{\bar{j}}\log\det\left(g_{p\bar{q}}\right). \tag{2.4}$$

The Ricci form is a representative of the manifold's First Chern Class, which we define to be $c_1(X)=-K_X$, the negative of the First Chern Class of the canonical line bundle of X. It is clear that the position of this class with respect to the Kähler cone determines whether a Kähler manifold can support a Kähler-Einstein metric of a particular sign.