Journal of Nonlinear Modeling and Analysis
Volume 6, Number 2, June 2024, 273–287
http://jnma-online.com
DOI:10.12150/jnma.2024.273

A Novel Numerical Simulations for Fornberg-Whitham and Modified Fornberg-Whitham Equations with Nonhomogeneous Boundary Conditions

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Abstract In this study, the numerical solutions of the Fornberg-Whitham (FW) equation modeling the qualitative behavior of wave refraction and the modified Fornberg-Whitham (mFW) equation describing the solitary wave and peakon waves with a discontinuous first derivative at the peak have been obtained. To obtain numerical results, the collocation finite element method has been combined with quintic B-spline bases. Although there are solutions to these equations by semi-analytical and analytical methods in the literature, there are very few studies using numerical methods. The stability analysis of the applied method is examined by the von-Neumann Fourier series method. We have considered four test problems with nonhomogeneous boundary conditions that have analytical solutions to show the performance of the method. The numerical results of the two problems are compared with some studies in the literature. Additionally, peakon wave solutions and some new numerical results of the mFW equation, which are not available in the literature, are given in the last two problems. No comparison has been made since there are no numerical results in the literature for the last two problems. The error norms L_2 and L_∞ are calculated to demonstrate the presented numerical scheme's accuracy and efficiency. The advantage of the scheme is that it produces accurate and reliable solutions even for modest values of space and time step lengths, rather than small values that cause excessive data storage in the computation process. In general, large step lengths in the space and time directions result in smaller matrices. This means less storage on the computer and results in faster outcomes. In addition, the present method gives more accurate results than some methods given in the literature.

Keywords Fornberg-Whitham Equation, modified Fornberg-Whitham equation, solitary waves, peakon waves, wind waves, quintic B-spline bases, collocation method

MSC(2010) 65M70, 35G31, 65Z05, 65D05, 65D07, 65L20.

1. Introduction

Numerical methods have become an essential tool for mathematicians and engineers in solving nonlinear partial differential equations in recent years. Traveling wave

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solutions constitute an important class of solutions of nonlinear partial differential equations. The Fornberg-Whitham equation, which has a traveling wave solution called the Kink-like wave solution and anti Kink-like solutions, is as follows [1]

$$U_t - U_{xxt} + U_x = UU_{xxx} - UU_x + 3U_x U_{xx}. \tag{1.1}$$

Whitham [2], who studied the qualitative behavior of waves, was the first to propose Eq. (1.1). Fornberg and Whitham [3] obtained a peakon wave solution $U(x,t) = Ae^{-\frac{1}{2}|x-4t/3|}$, where A is a constant. He et al. [4] gave some peakon and solitary wave solutions to the following modified Fornberg-Whitham equation, obtained by taking U^2U_x instead of the nonlinear term UU_x in (1.1):

$$U_t - U_{xxt} + U_x = UU_{xxx} - U^2U_x + 3U_x U_{xx}. \tag{1.2}$$

A notable feature of the mFW equation is that it generates peakon wave solutions. A peakon wave is a wave whose first derivative is discontinuous due to the peaks at its peak. Water covers 71 percent of the earth, and a significant portion of the sun's radiant energy that is not reflected into space is absorbed by the oceans' water. This absorbed energy heats the water, which heats the air above the oceans and creates air currents caused by differences in air temperature. These air currents create wind waves and return some energy to the water. Although the height of the wind waves varies, they reach the shore by traveling long distances [27].

Therefore, it has recently become reasonably interesting to solve numerically and analytically FW and mFW equations, which have many mathematical properties. Marasi and Aqdam [1] used the Homotopy-Pade technique to solve the FW equation. He et al. [4] investigated the mFW equation using bifurcation theory and phase portrait analysis, obtaining some peakons and solitary wave solutions. Dehghan and Heris [5] showed that the variational iteration method and the homotopy perturbation method are powerful and suitable methods for solving the FW equation. Lu [6] used a variational iteration method to solve the FW type equations. Boutarfa et al. [7] obtained the solutions of three types of the FW equations by applying the reproducing kernel Hilbert space method. Hörmann and Okamoto [8] studied spatially periodic solutions of the FW equation to illustrate the mechanism of wave breaking and the formation of shocks for a large class of initial data employing Godunov's finite difference method. Hesam et al. [9] presented a reduced differential transform method for solving FW type equations. Az-Zo'bi [10] implemented the simplest equation method to construct exact traveling-wave solutions to the mFW equation. Li and Song [11] studied the two-component FW equation and obtained the kink-like wave and compacton-like wave solutions. Zhou and Tian [12] utilized the bifurcation method to get traveling wave solutions called kink-like wave solutions and anti kink-like wave solutions for the FW equation. Chen et al. [13] obtained smooth, peaked, and cusped solitary wave solutions of the FW equation under inhomogeneous boundary conditions. Ramadan and Al-luhaibi [14] presented an approximate analytical solution of the nonlinear FW equation using the new iterative method. Abidi and Omrani [15] implemented the variational iteration method and homotopy-perturbation method to solve the nonlinear FW equation analytically. Chen et al. [16] gave some smooth periodic wave, smooth solitary wave, periodic cusp wave, and loop-soliton solutions of the FW equation, and they made some numerical simulations. Biazar and Eslami [17] proposed an analytical method for solving FW type equations based on the homotopy perturbation method. Abidi and Omrani [18] utilized the homotopy analysis method to