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Fractional Langevin Equation at Resonance*

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Abstract The study of fractional Langevin equation has obtained abundant results in recent years. However, there are few studies on resonant fractional Langevin equation. In this paper, we investigate boundary value problems for fractional Langevin equation at resonance. By virtue of Banach contraction mapping principle and Leray-Schauder fixed point theorem, we obtain the uniqueness and existence of solutions. In addition, we get different stability results, including Ulam-Hyres stability and generalized Ulam-Hyres stability. Finally, give relevant examples to demonstrate the main results.

Keywords Fractional order, Langevin equation, resonance, boundary value problems, fixed point theorem

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1. Introduction

Langevin equation is a motion equation proposed by Langevin in 1908 when he studied Brownian motion. It is well known that Langevin equation is widely used to describe the physical phenomena of waves in evolving environments [1,2]. However, in the complex evolution environment, Langevin equation of integer order cannot be described correctly. The fractional Langevin equation appears, that is, the integer derivative is replaced by the fractional derivative in the equation [3]. Compared with integer order, fractional calculus has non-local and memory properties and can better describe physical phenomena, such as heat conduction of soft matter, fractal phenomena and image processing [4,5].

The fractional Langevin equation is widely used in various fields [6–9]. In biochemistry, Langevin equation can be used to study protein folding dynamics [6]. In physics, Langevin equation can be used to study quantum noise [7]. For example, the stable two-state update model can be abstracted into the following fractional Langevin equation, which describes the underdiffusion process of potential fields and external signal forces [8]

$$ {}_0^cD_t^{\alpha}x(t) = ax - bx^3 + A\cos(2\pi ft) + \xi(t), \quad 0 < a < 1. $$

Based on the extensive application of the fractional Langevin equation, the solutions to its initial value problem and boundary value problem are concerned by

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scholars [10–15]. For example, Ahmad et al. [10] considered the existence and uniqueness of solutions for three-point boundary value problem of fractional Langevin equation

$$\begin{cases} {}^cD^{\beta}({}^cD^{\alpha}+\lambda)x(t)=f(t,x(t)), & 0<t<1, \\ x(0)=0, x(\eta)=0, x(1)=0, & 0<\alpha\leq 1, 1<\beta\leq 2. \end{cases}$$

Baleanu et al. [11] studied the existence of solutions for fractional Langevin equation involving Atangana-Baleanu operators

$$\begin{cases} {}^{ABD}D^{\beta}({}^{ABD}D^{\alpha}+\lambda)x(t)=f(t,x(t)), & 0<t<1, \\ x(0)=\gamma_1, x'(0)=\gamma_2, & 0<\alpha,\beta\leq 1. \end{cases}$$

Wang et al. [12] investigated the fractional Langevin equation with integro-differential strip-multi-point boundary conditions

$$\begin{cases} {}_0^{\beta}D^{\alpha}({}_0^{\gamma}D^{\alpha}+\lambda)x(t)=f(t,x(t),({}_0^{\gamma}D^{\alpha}+\lambda)x(t)), & 0<t\leq d, \\ t^{\alpha(1-\gamma)}x(t)|_{t=0}=\omega_1\int_0^{\eta}x(s)ds+\sum\limits_{i=1}^{m}\mu_i x(\xi_i), & 0<\alpha,\beta,\gamma<1, \\ t^{\alpha(1-\beta)}({}_0^{\gamma}D^{\alpha}+\lambda)x(t)|_{t=0}=\omega_2\int_0^{\eta}{}_0^{\gamma}D^{\alpha}x(s)ds+\sum\limits_{i=1}^{m}\nu_i{}_0^{\gamma}D^{\alpha}x(\xi_i). \end{cases}$$

For the study of fractional Langevin equation, as far as we know, it is considered in the case of non-resonance, and the resonance case has not been studied at present. Based on this, this paper establishes the fractional Langevin equation at resonance. Aiello [16] used the Langevin equation to describe the random movement of pollen under the action of water molecules. In the network public opinion environment, netizens' opinions are like the movement of pollen under the action of water molecules, which is very similar to the evolutionary structure of Langevin equation [17]. The resonance phenomenon between multiple events in network public opinions can be abstracted into the fractional Langevin equation at resonance, which makes the study of the resonant fractional Langevin equation not only have theoretical significance [18, 19], but also have certain practical value.

In addition to the existence and uniqueness of solutions, it is important to know the stability of them. As one of the qualitative theories of fractional differential equations, stability theory has gradually penetrated into various fields [20]. In the process of establishing the differential model, there are inevitably small disturbances that cannot be estimated, and these disturbances make the stability of differential equations fundamentally change. The stability of the system is an important basis to judge whether the system can run normally, so the stability of the system is worth further study. Antoniadou [21] proposed five different resonance orbitals, and found that the resonance orbitals with stability generally have larger eccentricity. Wang et al. [22] investigated the Ulam-Hyers stability of fractional Langevin equations

$$\begin{cases} {}^cD_t^{\beta}({}^cD_t^{\alpha}+\lambda)x(t)=f(t,x(t)), & t\in J', \\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=I_k, & I_k\in\mathbb{R}. \end{cases}$$

In this paper, we consider the following boundary value problem for fractional Langevin equation at resonance

$$\begin{cases} {}^cD_{0^+}^{\beta}({}^cD_{0^+}^{\alpha}-\lambda)x(t)=f(t,x(t)), & t\in(0,1), \\ x(0)=0, {}^cD_{0^+}^{\alpha}x(0)=0, x(1)=\eta x(\xi), \end{cases} \tag{1.1}$$