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Bifurcation of Limit Cycles of a Perturbed Pendulum Equation*

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Abstract This paper investigates the limit cycle bifurcation problem of the pendulum equation on the cylinder of the form $\dot{x} = y, \dot{y} = -\sin x$ under perturbations of polynomials of $\sin x$, $\cos x$ and y of degree n with a switching line $y = 0$. We first prove that the corresponding first order Melnikov functions can be expressed as combinations of anti-trigonometric functions and the complete elliptic functions of first and second kind with polynomial coefficients in both the oscillatory and rotary regions for arbitrary n. We also verify the independence of coefficients of these polynomials. Then, the lower bounds for the number of limit cycles are obtained using their asymptotic expansions with $n = 1, 2, 3$.

Keywords Pendulum equation, complete elliptic function, Melnikov function, limit cycle

MSC(2010) 34C08, 34C07.

1. Introduction and main results

Consider the following non-smooth near-integrable differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} p^+(x,y) + \varepsilon f^+(x,y) \\ q^+(x,y) + \varepsilon g^+(x,y) \end{pmatrix}, & y \geq 0, \\ \begin{pmatrix} p^-(x,y) + \varepsilon f^-(x,y) \\ q^-(x,y) + \varepsilon g^-(x,y) \end{pmatrix}, & y < 0, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and $p^{\pm}(x,y)$, $q^{\pm}(x,y)$, $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ are C^{∞} smooth functions. When $\varepsilon = 0$, system (1.1) is a non-smooth integrable differential equation and has a family of piecewise smooth closed orbits to form a generalized annulus (for short, period annulus). In recent years, the limit cycle bifurcation problems of system (1.1) have received considerable attention from mathematical scholars, and some important results have been obtained, when $p^{\pm}(x,y)$, $q^{\pm}(x,y)$, $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ are polynomials of x and y. See [3, 4, 7, 11–14, 18–21, 23, 24] and the references therein. For example, the authors in [7, 14] established a formula for the first order Melnikov function (called Melnikov function method), which

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plays a key role in studying the number of limit cycles of system (1.1). The authors in [4, 12, 13] developed the averaging method to non-smooth differential systems. Then the authors in [6, 10] showed the equivalence of these two methods. The authors in [20] studied the limit cycle problems of two kinds of quadratic reversible systems with non-smooth polynomial perturbation by using Picard-Fuchs equation.

But when $p^{\pm}(x,y)$, $q^{\pm}(x,y)$, $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ are not polynomials of x and y, such as trigonometric functions or trigonometric polynomials, there are a few results. For instance, the authors in [2] considered a pendulum-like equation of the form

$$\ddot{x} + \sin x = \varepsilon \sum_{s=0}^{m} Q_{n,s}(x)\dot{x}^s,$$

where $Q_{n,s}(x)$ are trigonometric polynomials of degree n, and got the upper bounds on the number of zeros of its associated first order Melnikov functions, in both the oscillatory and rotary regions. Another interesting perturbed whirling pendulum is the equation

$$\dot{x} = y,\ \ \dot{y} = \sin x(\cos x - r) + \varepsilon y(\cos x + a), \tag{1.2}$$

where a and $r \geq 0$ are real parameters, and $\varepsilon > 0$ is a small parameter, which was considered in [9]. The authors proved that, depending on the value of the parameter, the period function of system (1.2) is either monotone or has exactly one critical point using Picard-Fuchs equation method. By using the averaging method of first order, the authors in [1] obtained the exact number of limit cycles of the equation

$$\dot{x} = -y,\ \ \dot{y} = x + \varepsilon(1 + \cos^m(\theta))Q(x,y),$$

where $\varepsilon > 0$ is a small parameter, $\theta = \arctan(y/x)$ and $Q(x,y)$ is a polynomial of degree n. The non-smooth form of the above equation is

$$\dot{x} = -y,\ \ \dot{y} = x + \varepsilon(1 + \cos^m(\theta)) \sum_{k=1}^{2} \chi_{S_k}(x,y)Q_k(x,y),$$

where χ_S is the characteristic function of a set S, $Q_k(x,y)$ is a polynomial of degree n, and $S_1 = \{(x,y) : y \geq 0\}$ and $S_2 = \{(x,y) : y \leq 0\}$, considered by [16]. The authors got the exact number of limit cycles of this differential equation by using the averaging method of first order. Recently, the authors in [17] established some general methods on the existence of limit cycles bifurcating from closed orbits of a near-Hamiltonian system on the cylinder by the Melnikov function method and derived the expansions of the first order Melnikov function, which were used to consider the bifurcation problem of limit cycles near a double homoclinic loop.

In the current work, we will give the lower bounds of the number of limit cycles of the single pendulum

$$\dot{x} = y, \quad \dot{y} = -\sin x \tag{1.3}$$

under perturbation of polynomials of $\sin x$, $\cos x$ and y of degree n with the switching line $y = 0$. The pendulum equation (1.3) is a Hamiltonian system with total energy

$$H(x,y) = \frac{1}{2}y^2 - \cos x + 1, \tag{1.4}$$