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Long-Time Asymptotics of Complex mKdV Equation with Weighted Sobolev Initial Data*

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Abstract In this paper, we apply $\bar{\partial}$-steepest descent method to analyze the long-time asymptotics of complex mKdV equation with the initial value belonging to weighted Sobolev spaces. Firstly, the Cauchy problem of the complex mKdV equation is transformed into the corresponding Riemann-Hilbert problem on the basis of the Lax pair and the scattering data. Then the long-time asymptotics of complex mKdV equation is obtained by studying the solution of the Riemann-Hilbert problem.

Keywords Riemann-Hilbert problem, complex mKdV equation, $\bar{\partial}$-steepest descent method, long-time asymptotics

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1. Introduction

The study of nonlinear partial differential equations (NLPDEs) has played an important role in the development of science and technology. Until now, NLPDEs can be used to explain some complex physical phenomena, including mathematics, fluid mechanics, plasma physics, atmospheric oceans, aerodynamics, etc [2–9]. Nowadays, the inverse scattering transformation [10–13], Hirota bilinear method [14–16], Darboux transformation [17, 18] and so on are effective methods to solve NLPDEs. Especially, the inverse scattering transformation is the first method which was found and used to obtain the exact solution of the soliton equation. In the early 20th century, the solution of Riemann-Hilbert (RH) problem was developed and promoted [19, 20]. In 1993, Deift and Zhou proposed the famous nonlinear steepest descent method to analyze the long-time asymptotic behavior of integrable evolution equations. Deift and Zhou analyzed the long-time asymptotic behavior of the solution to the initial value problem of the famous mKdV equation and Schrödinger equation [21, 22]. Cuccagna studied the asymptotic stability of N-soliton solutions of the defocusing nonlinear schrödinger equation by $\bar{\partial}$-steepest descent method [23]. Robert analyzed the derivative nonlinear schrödinger equation via $\bar{\partial}$-steepest descent method [24]. In addition, Fan, Geng and Ma studied the soliton solutions and long-time asymptotic behavior of some integrable evolution equations based on

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RH problem [25–36]; among them, Ma has already done some work on nonlocal equations [35,36].

In this paper, we study the equation derived from the Lax pair given by Yishen Li [37]. The Lax pair is

$$\begin{aligned} &\psi_x = -i\lambda\sigma_3\psi + P\psi, \\ &\psi_t = (\zeta\lambda^3 + \eta\lambda^2 + \vartheta\lambda + \iota)\sigma_3\psi + Q\psi, \end{aligned} \tag{1.1}$$

where $\psi(x,t,\lambda)$ is a 2×2 matrix, σ_3=diag$(1,-1)$, and

$$P = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

$$\begin{aligned} Q = {}& i\zeta\lambda^2 P - i\lambda \begin{pmatrix} \frac{i\zeta}{2}uv & -\frac{i\zeta}{2}u_x - \eta u \\ \frac{i\zeta}{2}v_x - \eta v & -\frac{i\zeta}{2}uv \end{pmatrix} \\ &- \begin{pmatrix} \frac{i\zeta}{4}(uv_x - vu_x) - \frac{\eta}{2}uv & -\frac{i\zeta}{4}(-u_{xx} + 2u^2v) + \frac{\eta}{2}u_x - i\vartheta u \\ \frac{-i\zeta}{4}(-v_{xx} + 2uv^2) - \frac{\eta}{2}v_x - i\vartheta v & -\frac{i\zeta}{4}(uv_x - vu_x) + \frac{\eta}{2}uv \end{pmatrix}. \end{aligned} \tag{1.2}$$

The Lax pair (1.1) derives the following system:

$$\begin{cases} u_t = -\frac{i\zeta}{4}(u_{xxx} - 6uvu_x) - \frac{\eta}{2}(u_{xx} - 2u^2v) + i\vartheta u_x + 2\iota u, \\ v_t = -\frac{i\zeta}{4}(v_{xxx} - 6uvv_x) + \frac{\eta}{2}(v_{xx} - 2v^2u) + i\vartheta v_x - 2\iota v. \end{cases} \tag{1.3}$$

(I)Taking $\zeta = -4i$, $\eta = \vartheta = \iota = 0$, and $v = -1$, system (1.3) reduces to the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.4}$$

(II)Taking $\zeta = -4i$, $\eta = \vartheta = \iota = 0$, and $v = -u$, system (1.3) reduces to the mKdV equation:

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{1.5}$$

(III)Taking $\eta = -2i$, $\zeta = \vartheta = \iota = 0$, and $v = \mp\overline{u}$, system (1.3) reduces to the nonlinear Schrödinger equation:

$$iu_t + u_{xx} \pm 2u^2\overline{u} = 0, \tag{1.6}$$

where superscript bar denotes complex conjugate.

(IV)Taking $\iota = -2$, $\zeta = \vartheta = \iota = 0$, and $q_x = uv = \left(\frac{u_x}{u}\right)_x$, system (1.3) reduces to the Burger equation

$$q_t = 2qq_x - q_{xx}. \tag{1.7}$$

In addition, taking $\zeta = -i\alpha$ $(\alpha > 0)$, $\eta = \vartheta = \iota = 0$ and $v = \overline{u}$, system (1.3) reduces to the complex mKdV equation:

$$u_t = \frac{\alpha}{4}(-u_{xxx} + 6|u|^2u_x), \tag{1.8}$$

where $u(x,t)$ is complex-valued function of variate (x,t). In [38], Chen and Liu obtained the long-time asymptotics of the mKdV equation in weighted Sobolev spaces. However, the long-time asymptotics of the complex mKdV equation have