Unique Solution for a General Coupled System of Fractional Differential Equations*

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Abstract This paper discusses a new coupled system of Riemann-Liouville fractional differential equations, in which the nonlinear terms include the Riemann-Liouville fractional integrals and the boundary value problems involve three-points. We seek also for the existence and uniqueness of solutions for this new system. We first get some useful properties of the Green's function generated by the system, and then we apply a fixed point theorem of increasing φ-(h, e)-concave operators to this new coupled system. Finally, we gain the existence and uniqueness results of the solution for this problem. In the end, a concrete example is structured to illustrate the main result.

Keywords Existence and uniqueness, coupled system of fractional differential equations, fixed point theorem, φ-(h, e)-concave operators

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1. Introduction

This paper considers the existence and uniqueness of solutions for a new coupled system of fractional differential equations:

$$
\begin{cases}
D^\alpha u(t) + f(t, v(t), I^\gamma v(t)) = a, t \in (0, 1), \\
D^\beta v(t) + g(t, u(t), I^\delta u(t)) = b, t \in (0, 1), \\
u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \\
v(0) = v'(0) = 0, v'(1) = \mu v'(\xi),
\end{cases}
\tag{1.1}
$$

where $2 < \alpha, \beta \le 3, 1 < \gamma, \delta \le 2$, and $\alpha - \delta \ge 1, \beta - \gamma \ge 1, 0 < \xi < 1, 0 < \mu\xi^{\alpha-2} < 1$, $0 < \mu\xi^{\beta-2} < 1$, a, b are nonnegative constants. D^α and D^β are the standard Riemann-Liouville fractional derivatives, and $f, g : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous functions. We employ a fixed point theorem of increasing φ-(h, e)-concave operators by Zhai and Wang to study the system (1.1).

In recent decades, fractional calculus has been widely used as a tool to study many problems in different research fields, such as engineering, biology, physics,

and so on. Because the extensive development and application of fractional calculus equation theory, the theme of fractional differential equation system has developed into a crucial research field. Based on it, the theoretical research on the existence of solutions for fractional problems has attracted widespread attention. For some gained studies on fractional differential equations, we can refer to [3, 7–13, 19–23, 27–39] and others. Many problems of coupled systems involving fractional differential equations have been investigated extensively, see [9, 15, 22, 24] and others. From literature, we know that the mathematical definition of the differintegral operator of fractional order has been the subject of different approaches, and the most used are the Riemann-Liouville (RL), the Grünwald-Letnikov (GL), and the Caputo's (C) definitions. Although they are different in differential equations, some fixed methods can be used to study these problems and some similar results under different conditions are also obtained. Based upon this reason, here we consider (1.1) only involving Riemann-Liouville fractional derivative rather than other derivatives.

Nowadays, many researchers devoted themselves to determining the solvability of system of fractional differential equations with different boundary conditions, specifically to the study of existence of solutions to some systems of fractional differential equations, see [4–6, 9–14, 16–21, 23] for example and the references therein. The usual methods used are Schauder's fixed point theorem, Banach's fixed point theorem, Guo-Krasnoselskii's fixed point theorem on cone, monotone iterative technique, the method of lower and upper solutions, nonlinear differentiation of Leray-Schauder type and others. In [10], the authors considered a system of fractional differential equations:

$$
\begin{cases}
D^{\alpha} u(t) + f(t, v(t), D^{\gamma} v(t)) = 0, 0 < t < 1, \\
D^{\beta} v(t) + g(t, u(t), D^{\delta} u(t)) = 0, 0 < t < 1, \\
u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \\
v(0) = v'(0) = 0, v'(1) = \mu v'(\xi),
\end{cases}
$$

where, $2 < \alpha, \beta \le 3$, $1 < \gamma, \delta \le 2$, and $\alpha - \delta \ge 1$, $\beta - \gamma \ge 1$, $0 < \xi < 1$, $0 < \mu \xi^{\alpha-2} < 1$, $0 < \mu \xi^{\beta-2} < 1$. D^{α} and D^{β} are the Riemann-Liouville fractional derivatives, and $f, g : [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \to [0, +\infty)$ are continuous functions and depend on the unknown functions as well as their lower order fractional derivatives. The authors got the existence and uniqueness of positive solutions for the system by utilizing fixed point theorems due to Schauder and Banach.

This paper considers the existence and uniqueness of solutions for system (1.1). As is well known, no papers have considered the solutions of system (1.1) from literatures. So it is worthwhile to investigate (1.1).

2. Preliminaries and previous results

Definition 2.1. [1, 2] The Riemann-Liouville fractional integral of order $\alpha > 0$, for a function $f : (0, +\infty) \to (-\infty, +\infty)$, is given by

$$
I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds, t > 0,
$$

provided the integral exists.