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The First Eigenvalue of (p, q)-Laplacian System on C-Totally Real Submanifold in Sasakian Manifolds

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Abstract Consider (M, g) as an n-dimensional compact Riemannian manifold. Our main aim in this paper is to study the first eigenvalue of (p, q)-Laplacian system on C-totally real submanifold in Sasakian space of form $\bar{M}^{2m+1}(\kappa)$. Also in the case of $p, q > n$ we show that for $\lambda_{1,p,q}$ arbitrary large there exists a Riemannian metric of volume one conformal to the standard metric of S^n.

Keywords Eigenvalue, (p, q)-Laplacian system, geometric estimate, Sasakian manifolds

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1. Introduction

Studying the bounds of the eigenvalue of the Laplacian on a given manifold is a key aspect in Riemannian geometry. A major objective of this purpose is to study eigenvalue that appears as a solution of Dirichlet or Neumann boundary value problems for curvatures functions. By reason of the theory of self-adjoint operators, the spectral properties of linear Laplacian were studied extensively. As an example, mathematicians are generally attracted to the spectrum of the Laplacian on compact manifolds with or without boundary or on noncompact complete manifolds in these two cases the linear Laplacian can be uniquely extended to self-adjoint operators (see [7,8]).

Consider (M^n, g) as an n-dimensional compact Riemannian manifold. Let $u : M \longrightarrow \mathbb{R}$ be a smooth function on M or $u \in W^{1,p}(M)$ where $W^{1,p}(M)$ is the Sobolev space. The p-Laplacian of u for $1 < p < \infty$ is defined as

$$\begin{aligned}\Delta_p u &= div\left(|\nabla u|^{p-2}\nabla u\right)\\ &= |\nabla u|^{p-2}\Delta u + (p-2)\,|\nabla u|^{p-4}\,(Hess\,u)\,(\nabla u, \nabla u)\,,\end{aligned}$$

where

$$\begin{aligned}(Hess\,u)\,(X, Y) &= \nabla\,(\nabla u)\,(X, Y)\\ &= X\,(Yu) - (\nabla_X Y)\,u \qquad X, Y \in \chi\,(M)\,.\end{aligned}$$

The first eigenvalues of p-Laplace operator in both Dirichlet and Neumann cases have been studied in many papers (see for example [14]).

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In this paper we are going to study the first Dirichlet eigenvalue of the system

$$\begin{cases} \Delta_p u = -\lambda |u|^{\alpha} |v|^{\beta} v, & \text{in M}, \\ \Delta_q v = -\lambda |u|^{\alpha} |v|^{\beta} u, & \text{in M}, \\ u = v = 0, & \text{on } \partial\text{M}, \end{cases} \tag{1.1}$$

where $p, q > 1$ and α, β are real numbers such that

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

Let (M, g) be an n-dimensional compact Riemannian manifold. The first Dirichlet eigenvalue of system (1.1) is defined as

$$\lambda_{1,p,q}(M) = \inf_{u,v \neq 0} \Big\{ \frac{1}{\int_M |u|^{\alpha+1} |v|^{\beta+1} dv} \Big[\frac{\alpha+1}{p} \int_M |\nabla u|^p dv + \frac{\beta+1}{q} \int_M |\nabla v|^q dv \Big] \Big\},$$

where

$$(u, v) \in W_0^{1,p}(M) \times W_0^{1,q}(M) \setminus \{0\}.$$

As an example, the second author studied the first eigenvalue of the system (1.1) in [3].

The (p, q)-Laplacian system (1.1) was studied before in many papers. As an example the authors of this paper studied the first eigenvalue of the general case of the system (1.1) under the integral curvature condition in [9]. These types of systems have been found in different cases in physics. For example, they are relevant to the study of transport of electron temperature in a confined plasma and also to the study of electromagnetic phenomena in nonhomogeneous super conductors (see [5, 6]).

The study of submanifolds, especially Legendrian submanifolds in contact manifolds from the Riemannian geometric perspective was initiated in the 1970s. The main problem in this area is to establish the classes which include such submanifolds. As an example, nonharmonic biharmonic submanifolds in Sasakian space forms of low dimension were studied before in [11, 17]. The importance of studying eigenvalues of Laplacian was clearly obtained by Reilly in [15]. As a quick remark, we recall that the n-manifold M^n is called a minimal submanifold, if the mean curvature vector vanishes on M^n everywhere. In this case Reilly showed that the first eigenvalue of the Laplacian for a compact n-manifold isometrically immersed in Euclidean space is bounded above by n times the average value of the square of the norm of the mean curvature vector. Moreover, if the eigenvalue achieves this bound, then the submanifold is a minimal submanifold of some hypersphere in the Euclidean space. Ali et al. studied the first non-zero eigenvalue of p-Laplacian operator in [1].

Proposition 1.1. *Let Σ^n be an n-dimensional closed oriented C-totally real submanifold in a Sasakian space form $\bar{M}^{2m+1}(\kappa)$. The first non-zero eigenvalue $\lambda_{1,p}$ of the p-Laplacian satisfies the following conditions.*

- *If $1 < p \leq 2$ then*

$$\lambda_{1,p} \leq \frac{2^{\left(1-\frac{p}{2}\right)} (m+1)^{\left(1-\frac{p}{2}\right)} n^{\frac{p}{2}}}{(Vol(\Sigma))^{\frac{p}{2}}} \left(\int_{\Sigma^n} \left(\left(\frac{\kappa+3}{4} \right) + |H|^2 \right) dv \right)^{\frac{p}{2}}.$$