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Some Euler-Maclaurin-Type Inequalities by Means of Tempered Fractional Integrals

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Abstract This paper introduces an equality that is valid for tempered fractional integrals. By using this equality, we establish some Euler-Maclaurin-type inequalities for the case of differentiable convex functions involving tempered fractional integrals. Furthermore, our results are provided by using special cases of obtained theorems.

Keywords Euler-Maclaurin-type inequality, quadrature formulae, tempered fractional integrals, convex functions

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1. Introduction and preliminaries

The theory of inequalities is well-known and is still an active research field that has numerous practical applications in many mathematical areas. It is also well-known that inequality theory is closely related to the study of convex functions. Many famous inequalities, such as the Hermite–Hadamard-type inequalities Simpson, Newton, and Euler-Maclaurin-type inequalities, are established for convex functions. In addition, convexity is an significant tool in the study of inequalities, as it allows for the use of various analytical and geometric methods to prove inequalities. Finally, the study of convex functions and the theory of inequalities are deeply interconnected and complement one another.

Fractional calculus is related to inequality theory in that it provides a generalization of traditional calculus, allowing for the consideration of non-integer orders of differentiation and integration. These types of inequalities can be useful in the study of various mathematical and physical phenomena, such as the behavior of complex systems or the properties of certain functions. Moreover, fractional calculus has gained important attention from mathematicians in various fields of mathematics due to its essential characteristics and practical use in real-world problems. Due to the significance of fractional calculus, various mathematical researchers have proved numerous inequalities including fractional integrals. One can obtain the bounds of the new inequalities not only by using Hermite–Hadamard-type inequalities but also by using Simpson, Newton, and Euler–Maclaurin-type inequalities.

Definition 1.1 (See [12, 18]). The *Riemann–Liouville integrals* $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of

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order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b, \tag{1.2}$$

respectively. Here, $f \in L_1[a,b]$ and $\Gamma(\alpha)$ denotes the Gamma function defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du.$$

The fractional integral reduces to the classical integral for the case of $\alpha = 1$.

One can state the Simpson's rules in the following way:

i. The formula for Simpson's quadrature (also known as Simpson's 1/3 rule) is expressed as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{1.3}$$

ii. The formula for Simpson's second formula or Newton-Cotes quadrature formula (also known as Simpson's 3/8 rule (cf. [4])) is stated as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]. \tag{1.4}$$

iii. The corresponding dual Simpson's 3/8 formula - the Maclaurin rule based on the Maclaurin formula (cf. [4]) is created as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right]. \tag{1.5}$$

Formulae (1.3), (1.4), and (1.5) hold for every function f with continuous 4^{th} derivative on $[a,b]$.

The most popular Newton-Cotes quadrature containing three-point Simpson-type inequality is as follows.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a four times differentiable and continuous function on (a,b), and let $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Dragomir established an approximation of the remainder for Simpson's quadrature formula for bounded variation functions and the applications in theory of special means in paper [7]. Budak et al. proved some variants of Simpson-type inequalities for the case of differentiable convex functions by generalized fractional integrals in paper [1]. The reader is referred to [2, 13, 28, 30, 31] and the references therein for further information about fractional integrals.

The Classical closed-type quadrature rule is the Simpson 3/8 rule derived from the Simpson 3/8 inequality as follows: