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Controllability of Neutral Fractional Functional Differential Equations with Two Caputo Fractional Derivatives*

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Abstract Multiple fractional derivatives enrich the dynamic properties of fractional differential equations. This paper concerns with neutral fractional functional differential equations with two Caputo fractional derivatives. By using the fixed point methods with the fractional integral inequalities, the existence results and controllability of the equations are considered in the cases of finite delay and infinite delay, respectively. An example is given to illustrate the main results.

Keywords Neutral Caputo fractional functional differential equations, controllability, fractional integral inequalities, fixed point theorems

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1. Introduction

By using the fixed point method and other nonlinear analyses, the controllability of integer-order and fractional-order neutral functional differential equations is discussed widely, which involves Cauchy conditions and nonlocal conditions, abstract spaces and general spaces, non-impulsive and impulsive differential systems, as well as finite delay and infinite delay. At present, the approximate controllability [1, 2, 6, 11, 13, 15, 25, 27, 32, 35, 36, 38, 46, 60, 61], controllability [3–5, 8–10, 12, 18, 19, 21, 22, 24, 28–31, 33, 37, 39–45], exact controllability [17, 26], total controllability [43], numerical controllability [7, 23], relative controllability [16, 20], etc. al, of fractional differential equation with only one fractional derivative operator are widely considered. In [47], the author considered the existence results of impulsive Caputo fractional functional differential inclusions with variable times in the case of finite delay

$$\begin{cases} {}^CD^{\alpha}[{}^CD^{\beta}x(t) - g(t, x_t)] \in F(t, x_t), t \in J := [0, T]; \\ x(t^+) = I_k(x(t)),\ {}^CD^{\beta}x(t^+) = I_k^*(x(t)),\ t = \tau_k(x(t)); \\ x(t) = \phi(t),\ t \in [-\tau, 0];\ {}^CD^{\beta}x(0) = \mu \in R. \end{cases} \tag{1.1}$$

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In [48], the finite-time stability of the following Caputo fractional functional differential system is considered

$$\begin{cases} {}^cD_0^\alpha x(t) = Ax(t) + Bx(t-\tau(t)) + Dw(t) + f(t, x(t), x(t-\tau(t)), w(t)) \\ \qquad\qquad + {}^cD_0^\mu x(t-\tau(t)), t \in [0, T], \\ x(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{1.2}$$

In [49], using the fixed point method of multi-valued maps, the properties of semigroup and the properties of generalized Clarke's sub-differentials, the authors considered the existence of optimal feedback control for Caputo fractional neutral evolution systems in Hilbert spaces. In [50], using the fixed point method with theory of fractional calculus and stochastic analysis, the existence result for stochastic integro-differential equations in Hilbert spaces is obtained. Fore more details of the existence and stability of neutral fractional functional differential equations, see [6, 51–55, 57–60].

Taking into account the influence of multiple fractional derivatives of fractional differential equations, the authors considered the existence, attractivity and stability of fractional differential equations [61–66]. Till now, there have been few papers on the controllability of fractional differential equations with multiple fractional derivatives, especially in the case of finite time delay. Inspired by the above literature, in this paper, we investigate the following fractional functional differential equation with two Caputo fractional derivatives

$$\begin{cases} {}^CD^\alpha[{}^CD^\beta x(t) - g(t, x_t)] = f(t, x_t),\ t \in J := [0, T]; \\ x(t) = \phi(t),\ t \in I; \\ {}^CD^\beta x(0) = \mu \in R^n, \end{cases} \tag{1.3}$$

and the controllability form

$$\begin{cases} {}^CD^\alpha[{}^CD^\beta x(t) - g(t, x_t)] = f(t, x_t) + Bu(t),\ t \in J := [0, T]; \\ x(t) = \phi(t),\ t \in I; \\ {}^CD^\beta x(0) = \mu \in R^n, \end{cases} \tag{1.4}$$

where ${}^CD^\alpha$ and ${}^CD^\beta$ denote the Caputo fractional derivative of α, β order, respectively. $\alpha, \beta \in (0, 1), \alpha + \beta \in (1, 2)$; $x = (x_1, \cdots, x_n)^T$, $f, g \in C(J \times \mathcal{D}, R^n)$, $\phi \in \mathcal{D}, I = (-\infty, 0]$ or $[-\tau, 0]$. For any function x defined on $I \cup J$ and any $t \in J$, we denote by x_t the element of $\mathcal{D}$ defined by $x_t = x(t + \theta), \theta \in I$, which represents the history of the state from time $t - \tau$ up to the present time t. The norm of $\mathcal{D}$ is $\|u\|_{\mathcal{D}} := \sup\{\|u(t)\| : t \in I\}$. The control function $u(t) \in L^\infty(J, R^m)$ or $u(t) \in L^2(J, R^m)$, B is an $n \times m$ matrix.

In this article, some sufficient conditions for the existence results of (1.3) and controllability results of (1.4) are established, respectively, by using fractional integral inequalities, nonlinear analysis, fixed point approach including the contraction mapping principle and the Schaefer fixed point theorem.