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Uniqueness for the Semilinear Elliptic Problems*

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Abstract In this paper, we study the positive solutions of the semilinear elliptic equation

$$\begin{cases} Lu + g(x,u)u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, L is an elliptic operator, B is a general boundary operator and $g(\cdot,\cdot)$ is a continuous function. This is a general problem proposed by Amann [*Arch. Rational Mech. Anal.* 44 (1972)], Cac [*J. London Math. Soc.* 25 (1982)] and Hess [*Math. Z.* 154 (1977)]. We obtain various uniqueness results when the nonlinearity function g satisfies some additional conditions.

Keywords Elliptic, reaction-diffusion equation, uniqueness

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1. Introduction

In this paper, we study the positive solutions of the elliptic problem

$$\begin{cases} Lu + g(x,u)u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, the boundary $\partial\Omega$ consists of disjoint open subset Γ_0 and closed subset Γ_1 with finitely many components such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, and

$$L = -\sum_{i,j=1}^{N} a_{ij} D_i D_j + \sum_{i=1}^{N} a_i(x) D_i + a_0(x)$$

is a second order uniformly elliptic operator with $a_{ij}, a_i, a_0 \in C^{\mu}(\bar{\Omega})$, $i,j = 1,\cdots,N$. In (1.1), the boundary operator

$$B : C(\Gamma_0) \cup C^1(\Omega \cup \Gamma_1) \to C(\partial\Omega)$$

is given by

$$Bu = \begin{cases} u & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} + \beta(x)u & \text{on } \Gamma_1, \end{cases}$$

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where n is the outward unit normal of $\partial\Omega$, and $\beta \geq 0$ is continuous. Note that Γ_0 or Γ_1 can be empty. The operator B includes Dirichlet, Neumann as well as the Robin boundary condition. As far as the function $g(x, u)$, we assume that $g \in C(\Omega \times \mathbb{R}^+; \mathbb{R}^+)$ and there is a positive constant M such that

$$a_0(x) + g(x, M) \geq 0 \text{ in } \bar{\Omega}. \tag{1.2}$$

Throughout the paper, we consider the positive solution $\omega \in C^{2+\mu}(\Omega)$ of (1.1) with $0 \leq \omega(x) \leq M$ and $\omega(x) \not\equiv 0$ in $\bar{\Omega}$.

Note that (1.1) is widely used in the study of various diffusion problems and it is also called the diffusive logistic model [4, 5, 7, 8, 10, 12, 14–16]. In 1977, Hess [13] investigated the uniqueness problem on positive solutions of

$$\begin{cases} \Delta u = g(x, u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Since then, the positive solution problem (1.3) has attracted much attention, see Cac [9], Allegretto [1] and references therein. In this paper, we shall consider the uniqueness of positive solutions to the general problem (1.1). We shall obtain various uniqueness results under the assumptions that the nonlinear function g satisfies some additional conditions. In the case of (1.3), our results partially improve the classical conclusions of [1, 13]. We also prove the stability of positive solutions to (1.1).

Throughout this paper, let $\lambda[L; B, \Omega]$ be the principal eigenvalue of

$$\begin{cases} Lu = \lambda u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

associated with a positive eigenfunction $\phi(x)$. We then have

$$\frac{\partial \phi}{\partial n} < 0 \text{ for } x \in \Gamma_0,$$

and

$$\phi(x) > 0 \text{ for } x \in \Omega \cup \Gamma_1,$$

see [17, Chapter 8]. Moreover, we have

$$\lambda[L + s; B, \Omega] = \lambda[L; B, \Omega] + s$$

for all $s \in \mathbb{R}$ and

$$\lambda[L + f_1; B, \Omega] < \lambda[L + f_2; B, \Omega]$$

for any bounded functions $f_1 < f_2$ and $f_1 \not\equiv f_2$.

We first state our results on the existence of positive solutions to (1.1).

Theorem 1.1. *If $g(x, 0) < -\lambda[L; B, \Omega]$ for $x \in \bar{\Omega}$, then (1.1) admits a positive solution $\omega(x)$ satisfying*

$$\frac{\partial \omega}{\partial \nu} < 0 \textit{ for } x \in \Gamma_0,$$

and

$$\omega(x) > 0 \textit{ for } x \in \Omega \cup \Gamma_1.$$