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Pseudo-Differential Operators and $\mathfrak{T}$- Wigner Function on Locally Compact Communicative Hausdorff Groups

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Abstract In this article, we consider a harmonic analysis of locally compact groups and introduce a generalization of the classical cross-Wigner distribution defined on $G \times \hat{G}$ by

$$W_{\mathfrak{T}}(\psi, \varphi)(g,\ \xi) = \int_G \overline{\xi(h)} \psi(\tau_1(g,\ h)) \overline{\varphi(\tau_2(g,\ h))} d\mu(h).$$

We construct the so-called Weyl-Heisenberg frame on a locally compact communicative Hausdorff group and establish its properties. Thus, we show that assume Λ and Γ are closed cocompact subgroups of G and $\hat{G}$, respectively, then, for a given window $\phi \in L^2(G)$, either both systems $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ and $\{m_\kappa \tau_\upsilon \phi\}_{\kappa\in\Lambda^\perp,\ \upsilon\in\Gamma^\perp}$ are Gabor systems in $L^2(G)$, simultaneously, with the same upper bound, or neither $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ nor $\{m_\kappa \tau_\upsilon \phi\}_{\kappa\in\Lambda^\perp,\ \upsilon\in\Gamma^\perp}$ comprises a Gabor system. Finally, pseudo-differential operators on locally compact groups are studied, we establish that assuming a pseudo-differential operator A_a corresponds to the symbol $a \in W_{\tau, 1\circ\iota^{-1}}^{\infty,1}\left(G \times \hat{G}\right)$ then A_a is bounded operator $W_\tau^{p,q}(G) \to W_\tau^{p,q}(G)$.

Keywords Fourier transform, Wigner function, compact group, pseudo-differential operator, symbol, Weyl-Heisenberg frame

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1. Introduction (classical theory)

The concept of the pseudo-differential operator is a generalization of the partial differential operator. The theory of pseudo-differential operators is a fundamental tool of quantum physics and is widely interweaved in partial differential equations. The pioneer works belong to J. Kohn, L. Nirenberg, L. Hormander, E. Stein, and others [18, 26].

There are many articles dedicated to pseudo-differential operators and cross-Wigner functions for some recent issues that the reader may be interested in [1, 2, 11, 17, 21, 22]. In [7], authors consider Wigner analysis of linear operators, replacing standard Wigner function with the A-Wigner distribution with a symplectic matrix and developing a theory of global Hormander wavefront. The Gaussian state is

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considered [3] and a more complex system in [20]. The list of references consists of 26 articles.

The Weyl quantization is a correspondence between the set of pseudo-differential operators A_a closely and densely defined on the Hilbert space and the class of functions $a(x,z)$ mapping on the phase space $R^n \times R^n$. In its classical form [18], Weyl quantization is given by

$$A_a(f)(x) = \int_{R^n} a(x,z)\exp(2\pi i x\cdot z)\hat{f}(z)\,dz,$$

where the Fourier transform $\hat{f}$ of the function f is defined by

$$\hat{f}(x) = \int_{R^n} \exp(-2\pi i x\cdot z)\hat{f}(z)\,dz.$$

The Weyl quantization is intimately connected with the Wigner transform and cross-Wigner pseudo-distribution is defined by

$$W(f,g)(x,\ y) = \int_{R^n} \exp(-2\pi i y\cdot z) f\left(x+\frac{1}{2}z\right)\overline{g\left(x-\frac{1}{2}z\right)}dz$$

for all $f,g\in L^2(R^n)$.

The Weyl transform a_W of the symbol $a\in S'(R^n\times R^n)$ is given by

$$\langle a_W f, g\rangle = \langle a, W(f,g)\rangle$$

for all $f,g\in L^2(R^n)$. Then, the twisted product $a\#b$ of symbols a and b can be defined by

$$a\#b(x,\ y) = \int_{R^{4n}} a(z,u)\,b(v,w)\,\frac{\exp(4\pi i(x-z)(y-w))}{\exp(4\pi i(x-v)(y-u))}dzdudvdw.$$

Definition 1.1. The modulation space is a set $M_t^{\infty,1}(R^n\times R^n)$ of all functions $\sigma\in S'(R^n\times R^n)$ such that

$$\sup_{(x,p)\in R^n\times R^n}\left|\left(1+|x|^2+|p|^2\right)^{\frac{t}{2}} W(\sigma,\upsilon)(x,p,y,q)\right| \in L^1(R^n\times R^n)$$

for every window $\upsilon \in S'(R^n\times R^n)$. The space $M_0^{\infty,1}(R^n\times R^n) \equiv M^{\infty,1}(R^n\times R^n)$ is called a Sjostrand class [17].

Harmonic analysis shows that the modulation spaces $M^{\infty,1}(R^n\times R^n)$ constitute a Banach algebra with respect to the twisted product.

In the present article, we extend the ideas of classical harmonic analysis and the theory of pseudo-differential operators to locally compact Hausdorff groups [18]. To reach this goal we modernize methods of phase-time analysis and employ methods of convex estimations on Banach spaces. We redefine the $\mathfrak{T}$- Wigner pseudo-distribution by

$$W_{\Im}(\psi,\varphi)(g,\ \xi) = \int_G \overline{\xi(h)}\psi(\tau_1(g,\ h))\overline{\varphi(\tau_2(g,\ h))}d\mu(h)$$

for all $g\in G$ and all $\xi\in\hat{G}$, which satisfies the Plancherel formula $\|W_{\Im}(\psi,\varphi)\|_{L^2(G\times\hat{G})} \le const\,\|\psi\|_{L^2(G)}\|\varphi\|_{L^2(G)}$ for all $\psi,\varphi\in L^2(G)$. We develop a theory of modulation spaces $M_m^{p,q}(G)$ on locally compact commutative groups. As an interesting example, we establish that the Richczek operator A_a with the symbol $a\in W_{\tau,1\circ\iota^{-1}}^{\infty,1}\left(G\times\hat{G}\right)$ is a bounded linear operator from $W_\tau^{p,q}(G)$ to $W_\tau^{p,q}(G)$.