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On Dual K-g-Bessel Sequences and K-g-Orthonormal Bases*

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Abstract In Hilbert spaces, K-g-frames are an advanced version of g-frames that enable the reconstruction of objects from the range of a bounded linear operator K. This research investigates K-g-frames in Hilbert space. Firstly, using the g-preframe operators, we characterize the dual K-g-Bessel sequence of a K-g frame. We provide additional requirements that must be met for the sum of a given K-g-frame and its dual K-g-Bessel sequence to be a K-g-frame. At the end of this paper, we present the concept of K-g-orthonormal bases and explain their link to g-orthonormal bases in Hilbert space. We also provide an alternative definition of K-g-Riesz bases using K-g-orthonormal bases. This gives a better understanding of the concept.

Keywords K-g-frames, dual K-g-Bessel sequences, K-g-orthonormal bases, K-g-Riesz bases

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1. Introduction

In 1952 [9], Duffin and Schaeffer first introduced frames for Hilbert spaces to study problems in the nonharmonic Fourier series. Nowadays, frame theory has become widely used in various fields, such as filter theory [4], signal and image processing [5], encoding and transmission [15] and so on. For further information on frame theory, please refer to the literature [1, 6, 14].

The use of frame theory in Hilbert spaces has led to the emergence of many generalized frames. In 2012, Găvruta [11] introduced K-frames to study atomic systems. Unlike general frames, K-frames are limited to the range of a specific bounded linear operator K, making them more practical and flexible. Furthermore, studying bounded linear operators offers a new research approach. In 2006, Sun [19] introduced the concepts of g-Riesz bases and g-frames. Later, Xiao et al. [22] put forward the concept of K-g-frame, which was limited to the range of a bounded linear operator in Hilbert space and has gained greater flexibility in practical application relative to g-frame (see [2, 17]). K-g-frame, as a more general frame than

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g-frame and K-frame in Hilbert space, has become one of the most active fields in frame theory in recent years. K-g-frames are a generalization of g-frames, and many of their properties are similar to those of g-frames. However, there are some differences, which have led to K-g-frames becoming one of the most active fields in frame theory in recent years. Many studies have been conducted in this area, such as [8, 16, 18, 23]. No one has discussed g-orthonormal bases of range K. Moreover, many problems of K-g-frames, such as how to find the dual, have not been studied. Since the frame operator may not be invertible, there is no classical canonical dual for a K-g-frame, which partly causes the fact that there are few results on the duals of a K-g-frame. This motivates us in this paper to examine the duals of K-g-frames in greater detail and provide characterizations of K-g-orthonormal bases, which we have defined.

This article is divided into four sections, each with its own outline. Section 2 will review essential results related to g-frames and K-g-frames for Hilbert spaces. Moving on to Section 3, we will characterize the dual K-g-Bessel sequence of a K-g-frame by using the g-preframe operators. We will also give some conditions under which the sum of a given K-g-frame and its dual K-g-Bessel sequence is a K-g-frame, taking into account the g-preframe operators. In the final section, Section 4, we will study K-g-orthonormal bases and explore their relationship with g-orthonormal bases. Additionally, we will provide an equivalent characterization of K-g-Riesz bases by K-g-orthonormal bases and discuss their properties.

Throughout this paper, $\mathcal{M}$ and $\mathcal{N}$ are separable Hilbert spaces, and I is the identity operator on $\mathcal{M}$. $\mathbb{L}$ represents a countable index set. ONB denotes the orthonormal basis. Let $\mathcal{B}(\mathcal{M},\mathcal{N})$ be the space of all the bounded linear operators from $\mathcal{M}$ to $\mathcal{N}$ and write $\mathcal{B}(\mathcal{M}) = \mathcal{B}(\mathcal{M},\mathcal{M})$. For an operator $S \in \mathcal{B}(\mathcal{M},\mathcal{N})$, let ran$S$, ker$S$ and S^* be the range space, the nullspace and the adjoint of S, respectively. For a sequence of Hilbert spaces $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, $\left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$ is defined by

$$\left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2} = \left\{\{\xi_l\}_{l\in\mathbb{L}} : \xi_l \in \mathcal{M}_l, \sum_{l\in\mathbb{L}} \|\xi_l\|^2 < \infty\right\}.$$

2. Preliminaries

Here, we will review some key definitions and lemmas that will be required later.

Definition 2.1 ([19]). A sequence $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ is called a g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, if there exist two positive constants a and b such that

$$a\|\xi\|^2 \le \sum_{l\in\mathbb{L}} \|\mathcal{D}_l\xi\|^2 \le b\|\xi\|^2, \quad \xi \in \mathcal{M}. \tag{2.1}$$

We call a and b the lower and upper g-frame bounds, respectively. If only the right hand inequality of (2.1) holds, we call $\mathcal{D}$ a g-Bessel sequence for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$ with Bessel bound b. If $a = b$, we call $\mathcal{D}$ a tight g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, and if $a = b = 1$, we call $\mathcal{D}$ a Parseval g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$.

For a g-Bessel sequence $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, $T_{\mathcal{D}} : \mathcal{M} \to \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$ defines a bounded linear operator,

$$T_{\mathcal{D}}\xi = \{\mathcal{D}_l\xi\}_{l\in\mathbb{L}}, \forall\xi \in \mathcal{M}. \tag{2.2}$$