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Solution of Some Non-homogeneous Fractional Integral Equations by Aboodh Transform

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Abstract The present paper introduces the Aboodh transform technique as a method for obtaining solutions to a class of non-homogeneous fractional integral equations. The emphasis is placed on equations characterized by expressions involving Riemann-Liouville fractional integrals of orders $1, \frac{1}{2}$, and $\frac{1}{3}$. The paper includes illustrative examples that demonstrate the application of the Aboodh transform technique. These examples elucidate how the technique can effectively yield solutions for specific instances of the mentioned equations. The obtained solutions are presented in the form of Mellin-Ross functions.

Keywords Fractional differential equations, Aboodh transform, fractional integral equations, Mellin-Ross function, Riemann-Liouville fractional integral

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1. Introduction

Fractional calculus is indeed a mathematical theory that deals with derivatives and integrals of arbitrary complex or real order. Its origins can be traced back to the early 18^{th} century when mathematicians like Leibniz and L'Hopital started to investigate the meaning of fractional derivatives.

In particular, in 1695, L'Hopital posed the problem of finding the meaning of the derivative of order $n = 1/2, i.e., d^n y/dx^n$, and asked Leibniz for a solution. Leibniz himself was intrigued by the problem and tried to find a way to define fractional derivatives and integrals. However, it was not until the 19^{th} century that the concept of fractional calculus was fully developed by mathematicians like Liouville, Riemann, and Grunwald. They introduced the concept of fractional derivatives and integrals as a natural extension of the classical calculus. Since then, fractional calculus has found numerous applications in various fields, including engineering, economics, physics, and biology [3,5,6,15,19,20,22,23,29,37,38]. It has proven to be a powerful tool for modeling and analyzing complex systems with non-local and non-linear behavior, such as fractional differential equations.

Fractional derivatives, a fundamental component of fractional calculus, have garnered significant attention in recent years. They play a crucial role in modeling phenomena across various branches of engineering and science when dealing with

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real-world problems. Fractional calculus has facilitated the development of mathematical models for practical issues encountered in diverse fields such as dielectric polarization, viscoelasticity, electromagnetic waves, and electrode-electrolyte polarization [7–9, 11, 12, 16–18, 21, 31–35]. These applications highlight the practical relevance and broad impact of fractional calculus in addressing complex phenomena in engineering and scientific domains.

The Aboodh transform is a mathematical tool used to solve fractional differential equations. It is based on the fractional derivative of the generalized Mittag-Leffler function, which is a special function that arises in the study of fractional calculus [1,2,4,10,29,30]. The application of the Aboodh transform to non-homogeneous fractional integral equations is a relatively new area of research, and there is still much to be discovered and understood about its potential applications and limitations. However, preliminary findings suggest that it has the potential to be a powerful tool for solving a wide range of problems in fractional calculus and related fields.

In 2005, T. Morita [26] conducted a study on the initial value problem of fractional differential equations, employing the Laplace transform. In his work, he derived solutions for fractional differential equations involving the Riemann-Liouville fractional derivative as well as the Caputo fractional derivative or its modified form. Morita's research focused on obtaining solutions to these equations by utilizing the Laplace transform technique.

In 2010, T. Morita and K. Sato [27] conducted a study on the initial value problem of fractional differential equations with constant coefficients. The equations they considered were of the following forms:

$$ {}_0D_t^{\zeta} u(t) + l \cdot u(t) = f(t), $$
$$ {}_0D_t^{\zeta} u(t) + k \cdot \ {}_0D_t^{\xi} u(t) + l \cdot u(t) = f(t), $$

and

$$ {}_0D_t^{\gamma_n} u(t) + \sum_{r=0}^{n-1} l_r \cdot \ {}_0D_t^{\gamma_r} u(t) = f(t). $$

In these equations, ${}_0D_t^{\gamma_n}$ represents the (RL) fractional derivative, l_r are constants for $r = 0, 1, 2, 3, ..., n-1$, and $t \in \mathbb{R}^+$. Morita and Sato obtained solutions to these equations using techniques involving Green's function and distribution theory. Furthermore, they also studied the solution of a fractional differential equation of the form:

$$ (j_2 t + k_2)_0 D_t^{2\gamma} u(t) + (j_1 t + k_1)_0 D_t^{\gamma} u(t) + (j_0 t + k_0) u(t) = f(t), $$

where $\gamma = \frac{1}{2}, \gamma = 1, t \in \mathbb{R}^+$ and j_r, k_r are constants for r=0,1,2,3. For more details, refer to [28].

In 2018, C. Li [24] conducted a study on Abel's integral equation of the second kind. The equation is given by:

$$ y(t) + \frac{\lambda}{\Gamma(\gamma)} \int_0^t (t-\phi)^{\gamma-1} y(\phi) d\phi = f(t), \quad t > 0. \tag{1.1} $$

Here, Γ is the gamma function,$\gamma \in \mathbb{R}$, and λ is a constant . Equation (1.1) can be written in the form

$$ (1 + \lambda I_{0+}^{\gamma}) y(t) = f(t). $$