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On Some Relations of *R*-Projective Curvature Tensor in Recurrent Finsler Space

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Abstract In this paper, we present a novel class of relations and investigate the connection between the R-projective curvature tensor and other tensors of Finsler space F_n. This space is characterized by the property for Cartan's the third curvature tensor R^i_{jkh} which satisfies the certain relationship with given covariant vectors field, as follows:

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} = a_{lmn}R^i_{jkh} + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}) - 2[c_{lm}\mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})y^r$$

$$+ d_{ln}\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r + \mu_l\ \mathcal{B}_n\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r],$$

where $R^i_{jkh} \neq 0$ and $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$ is the Berwald's third order covariant derivative with respect to x^l, x^m and x^n respectively. The quantities $a_{lmn} = \mathcal{B}_n u_{lm} + u_{lm}\ \lambda_n$, $b_{lmn} = \mathcal{B}_n v_{lm} +\ u_{lm}\ \mu_n$, $c_{lm}\ = v_{lm}$, and $d_{ln} = \mathcal{B}_n \mu_l$ are non-zero covariant vector fields. We define this space a generalized $\mathcal{B}R$-$3rd$ recurrent space and denote it briefly by$G\mathcal{B}R$-$3RF_n$. This paper aims to derive the third-order Berwald covariant derivatives of the torsion tensor H^i_{kh} and the deviation tensor H^i_h. Additionally, it demonstrates that the curvature vector K_j, the curvature vector H_k , and the curvature scalar H are all non-vanishing within the considered space. We have some relations between Cartan's third curvature tensor R^i_{jkh} and some tensors that exhibit self-similarity under specific conditions. Furthermore, we have established the necessary and sufficient conditions for certain tensors in this space to have equal third-order Berwald covariant derivatives with their lower-order counterparts.

Keywords n-dimensional Finsler space F_n, generalized $\mathcal{B}R$-$3rd$ recurrent spaces, employing Berwald's third order covariant derivative, R^i_{jkh} Cartan's third curvature tensor

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1. Introduction

The study of recurrent Finsler spaces began in 1973 with the work of Sinha and Singh [24], who explored the properties of recurrent tensors in these spaces. The differential geometry of Finsler spaces subsequent research on recurrent Finsler spaces

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was conducted by Rund [20] in 1959 and 1981. While Abdallah [3] and Baleedi [15] in 2017, investigated the recurrence of Berwald's curvature tensors R^i_{jkh} and K^i_{jkh}. Building upon these foundational works, Ahsan and Ali [4] in 2014, studied the properties of W-curvature tensor. Opondo [18] and Abu-Donia et al. [10] introduced and analyzed the recurrence conditions of the curvature tensor W^i_{jkh} using Berwald's approach.

From 2019 to 2023, Ali et al. [11–13] and Shaikh et al. [21, 22] presented some properties of the tensors W and M. They delved into the semi-conformal symmetry a new symmetry of the spacetime manifold of the general relativity. Qasem and Abdallah [19] furthered this research by defining the generalized $\mathcal{B}$R-recurrent Finsler space and establishing the necessary and sufficient conditions for both the Berwald curvature tensor and Cartan's fourth curvature tensor to exhibit generalized recurrence. Subsequently, Al-Qashbari and Qasem [5] investigated generalized $\mathcal{B}$R-trirecurrent Finsler spaces. Then in 2020, Al-Qashbari [6–8] derived various identities for generalized curvature tensors in $\mathcal{B}$-recurrent Finsler spaces and other tensors.

The most recent contribution to this field is the work of Al-Qashbari and Al-Maisary [9], who studied generalized BW-fourth recurrent Finsler spaces in 2023. Chen, Decu et al. [16, 17] in 2021, introduced the concept of classification of Roter type spacetimes and recent developments in Wintgen inequality and Wintgen ideal submanifolds. In 2021 and 2022, Atashafrouz et al. [1] and Saleem et al. [23] studied the notions of D-recurrent Finsler metrics and the U-recurrent Finsler space respectively. Recently, Abdallah [2] studied the relationships between two curvature tensors in Finsler space. Embarking on an exploration of the inherent attributes of an n-dimensional Finsler space F_n, we presuppose that its metric function F adheres to the well-defined stipulations outlined in [18].

1. Positively homogeneous: $F(x, ky) = k\ F(x, y)$, $k > 0$.
2. Positively: $F\,(x, y) > 0 \quad , \quad y \neq 0$.
3. $\{\ \dot{\partial}_i\ \dot{\partial}_j F^2(x, y)\ \}\ \xi^i\ \xi^j$, $\ \dot{\partial}_i = \frac{\partial}{\partial y^i}$ is the positive definite for all variables ξ^i.

The corresponding metric tenser denoted by g_{ij}, the connection coefficients of Cartan represented by Γ^{*i}_{jk} and the connection coefficients of Berwald designated by G^i_{jk}, are all related to the metric function F.

$$(a)\ \ g_{ij}\ y^i\ y^j = F^2,\ (b)\ g_{ij}\ y^j = y_i,\ (c)\ g_{ij} = \frac{1}{2}\ \dot{\partial}_i y_j,\ (d)\ y_i\ y^{\ i} = F^2,$$
$$(e)\ g_{ij}\ g^{ik} = \delta^k_j = \{\begin{matrix} 1 & if\ \ j = k \\ 0 & if\ \ j \neq k \end{matrix}\ ,\ (f)\quad \delta^i_h\ g_{ik} = g_{hk}, \tag{1.1}$$
$$(g)\qquad \delta^i_k\ y^k = y^i,\ and\ (h)\ \ \delta^i_i = n.$$

The torsion tensor C_{ijk} is defined by [20]

$$C_{ijk} = \frac{1}{2}\ \dot{\partial}_i\ g_{jk} = \frac{1}{4}\ \dot{\partial}_i\ \dot{\partial}_j\ \dot{\partial}_k\ F^2, \tag{1.2}$$

and its associate is the torsion tensor C^i_{jk} which is defined by:

$$(a)\ C^h_{ik} = g^{hj}\ C_{ijk}\ ,\ (b)\ \ C^i_{jk}\ \ y^k = C^i_{kj}\ \ y^k = 0. \tag{1.3}$$