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Some New Discrete Hermite-Hadamard Inequalities and Their Generalizations

Xiaoyue Han[1] and Run Xu[2,†]

Abstract This article mainly studies some new discrete Hermite-Hadamard inequalities for integer order and fractional order. For this purpose, the definitions of h-convexity and preinvexity for a real-valued function f defined on a set of integers $\mathbb{Z}$ are introduced. Under these two new definitions, some new discrete Hermite-Hadamard inequalities for integer order related to the endpoints and the midpoint $\frac{a+b}{2}$ based on the substitution rules are proposed, and they are generalized to fractional order forms. In addition, for the h-convex function on the time scale $\mathbb{Z}$, two new discrete Hermite-Hadamard inequalities for integer order by dividing the time scale differently are obtained.

Keywords Discrete fractional calculus, h-convex functions, preinvex functions, Hermite-Hadamard inequalities, times scales

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1. Introduction

Convex theory has always been an important component of mathematical theory, which is often used to solve many problems in economics, optimization, engineering, and other fields [12,24,25]. Scholars obtained many classical inequalities by utilizing the different convexities of functions, such as Schur inequalities, Hermite-Hadamard (H-H) inequalities, and Ostrowski type inequalities [8,28,31].

Currently, many scholars are committed to studying the H-H inequality. In 1893, Hermite and Hadamard [14] proved the classical H-H inequality first, which gave estimates of the upper and lower bounds of the integral mean of any convex function, as follows:

If $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function in I and $u, v \in I$, where $u \le v$, then

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v f(\delta)\mathrm{d}\delta \le \frac{f(u)+f(v)}{2}. \tag{1.1}$$

In 2015, Noor et al. [22] obtained a new H-H integral inequality for h-convex

†the corresponding author.
Email address:xiaoyue991231@163.com(X. Han), xurun2005@163.com(R. Xu)
[1]School of Mathematical Sciences, Yangzhou University, 225002 Yangzhou, China
[2]School of Mathematical Sciences, Qufu Normal University, 273165 Qufu, China

functions by inserting a segmentation point $\frac{u+v}{2}$ on interval $[u,v]$:

$$\begin{aligned}\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}f\left(\frac{u+v}{2}\right) \leq & \frac{1}{4h\left(\frac{1}{2}\right)}\left[f\left(\frac{3u+v}{4}\right)+f\left(\frac{u+3v}{4}\right)\right] \leq \frac{1}{v-u}\int_u^v f(\delta)\mathrm{d}\delta \\ \leq & \left[\frac{f(u)+f(v)}{2}+f\left(\frac{u+v}{2}\right)\right]\int_0^1 h(t)\mathrm{d}t \\ \leq & \left\{[f(u)+f(v)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(t)\mathrm{d}t.\end{aligned}$$

It is an important method to generalize H-H inequality by dividing intervals differently. More researches on this aspect can be found in references [10, 30, 32].

Due to the importance of fractional operators in pure mathematics and applied mathematics, scholars defined various fractional integral operators from different directions. It has become one of the hot research topics to establish and study H-H inequalities based on generalized fractional order integrals.

In 2013, Sarikaya et al. [26] extended inequality (1.1) using Riemann-Liouville (R-L) fractional order integrals as follows:

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[J_{u^+}^\alpha f(v)+J_{v^-}^\alpha f(u)\right] \leq \frac{f(u)+f(v)}{2}.$$

In 2014, Sarikaya et al. [27] established a midpoint type H-H inequality for R-L fractional order integrals based on the midpoint of the interval $[u,v]$:

$$f\left(\frac{u+v}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^\alpha}\left[J_{\left(\frac{u+v}{2}\right)^+}^\alpha f(v)+J_{\left(\frac{u+v}{2}\right)^-}^\alpha f(u)\right] \leq \frac{f(u)+f(v)}{2}.$$

In 2017, Agarwal et al. [3] established some H-H inequalities via generalized k-fractional integrals. In 2020, Mehreen et al. [21] established H-H inequalities for p-convex functions via conformable fractional integrals. In 2023, Tariq et al. [33] presented a new version of the H-H inequalities for preinvex functions via non-conformable fractional integrals. For more research on fractional order H-H integral inequalities, please refer to references [7, 16, 18, 29, 36].

The theory of dynamic equations on time scales is a new field in mathematical science. In the past few years, some integral inequalities used for dynamic equations on time scales have attracted the attention of many scholars. Discrete calculus is a calculus theory on time scales, which is of great significance for describing the discontinuity of certain time variables. In recent years, with the development of discrete calculus, research on H-H inequalities for integer order and fractional order has gradually increased.

In 2016, Atıcı and Yaldız [2] defined the convexity of real functions on any time scale, and established discrete H-H inequalities for integer order and fractional order via convex functions on $\mathbb{Z}$. The main results are as follows:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{2(v-u)}\left[\int_u^v f(\xi)\Delta\xi+\int_u^v f(\xi)\nabla\xi\right] \leq \frac{f(u)+f(v)}{2},$$

and

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\varepsilon)}{2\Lambda(v-u)}\left[\Delta_{v-1}^{-\varepsilon}f(u-\varepsilon)+{}_{u+1}\nabla^{-\varepsilon}f(v)\right] \leq \frac{f(u)+f(v)}{2},$$