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Enhancing Jerk Nonlinear Equation by Adomain Decomposition Method

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Abstract In this manuscript, an approximate periodic solution of a nonlinear Jerk equation concerning the third order time derivative (ODE) is demonstrated. This method succeeds highly in constructing a periodic solution for Jerk nonlinear differential equations. The results investigated from this method are compared with those determined from other analytical and numerical methods. Our results demonstrate a significant consistency with the numerical solution in addition to the analytical methods.

Keywords Adomian decomposition method, nonlinear Jerk equation, high order differential equation

MSC(2010) 34A34 , 70Kxx.

1. Introduction

Jerk equation (JE) expresses the acceleration change with time; JE is the first derivative of acceleration (j) and can be expressed as follows.

$$\vec{j} = \frac{d\vec{a}}{dt} = \frac{d^2\vec{v}}{dt^2} = \frac{d^3\vec{s}}{dt^3}. \tag{1.1}$$

Having that
$\vec{a}$ is acceleration;
$\vec{v}$ is velocity;
$\vec{s}$ is displacement;
t is time.
As JE is a third derivative of displacement, differential equation (DE) of the form

$$J(\dddot{x}, \ddot{x}, \dot{x}, x) = 0 \tag{1.2}$$

Known as JE. It was found that a JE, which describes a system formed of three first-order, ordinary non-linear DE, is the minimal expression showing a chaotic

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manner [1]. The familiar form of the JE that keeps its style upon time and space reversal, and with the cubic non-linearities is determined as:

$$\dddot{x} = -\gamma\dot{x} - \alpha\dot{x}^3 - \beta x^2\dot{x} + \delta x\dot{x}\ddot{x} - \epsilon\dot{x}\ddot{x}^2, \tag{1.3}$$

where the parameters ϵ, δ, β, α and γ are constant. Here, for special cases and to compare with other studies at least one of the parameters δ, β and ϵ should be not equal to zero. The initial conditions (IC) are [1]

$$x(0) = 0, \quad \dot{x}(0) = B \neq 0, \quad \ddot{x}(0) = 0. \tag{1.4}$$

JE spreads widely in several mechanics and acoustics applications. Therefore, many methods are demonstrated to investigate an exact or approximate solution for it. Modified harmonic balance method (MHBM) [2] is used to find a 2^{nd} approximate solution of non-linear, third-order differential JE, but it is successful for a limited number of other non-linear equations. Hybrid block method (HBM) [3] solves JE with one hybrid point that is indicated to enhance effectively the local truncation errors of the basic formulas of the solutions and the derivatives at the end of the block. The Homotopy method, considered as one of the powerful methods in many physical and mechanical approximations [4], is extended to provide periodic solutions of a nonlinear jerk equation involving the 3^{rd} order time derivative. A suitable parameter is introduced to provide a numerical solution that proves well for the first few terms. The Analytical solutions [5], by Ramose, are developed for nonlinear, 3^{rd} order nonlinear DE to investigate a convergent solution. The solutions are determined as functions of the coefficients that concern the terms that are linearly proportional to the velocity and nonlinear terms, using means of transformations for periodic and non-periodic behavior. In our study, we use the LDM. The LDM is an effective, and accurate method for demonstrating analytical solutions to nonlinear problems. Also, it competes highly with other methods in finding accurate or approximate solutions for different nonlinear equations of physical or mechanical applications [6]. On the other hand, LDM proves the highest advantage in exceeding the difficulty that appears in the calculation of the Adomain polynomials, compared with other methods. The LDM divides the given equation into non-linear and linear parts, next obtains the highest-order derivative operator incorporated with the linear operator on each side, then provides Adomian's polynomials, and eventually investigates the series successive terms of the solution by a frequent relation using Adomian's polynomials. [7].

$$Lu + Nu + Ru = g \tag{1.5}$$

where L is an operator representing the highest order derivative; R is a linear operator for the remainder of the linear portion; N(u) represents the non-linear terms; lastly, g is the source term. This article is divided into three sections: the current section which demonstrates the introduction; the second section which demonstrates the mathematical analysis and results; and lastly, the conclusion in section 3.

2. Mathematical analysis and results

With appreciation of the general case, let us consider the case for $\alpha = \beta = 1$ and $\gamma = \delta = \epsilon = 0$. So, Eq.(1.3) becomes

$$\dddot{x} = -\dot{x}^3 - x^2\dot{x}. \tag{2.1}$$

We define a linear operator

$$L = \frac{d^3}{dt^3} \tag{2.2}$$

and the inverse operator

$$L^{-1} = \int\int\int dtdtdt. \tag{2.3}$$

In operator form, Eq.(2.1) can be displayed as

$$Lx = -\dot{x}^3 - x^2\dot{x}. \tag{2.4}$$

Applying L^{-1} to Eq.(2.4),

$$L^{-1}Lx = -L^{-1}\dot{x}^3 - L^{-1}x^2\dot{x}. \tag{2.5}$$

Depending on the IC as in Eq.(1.4) ,

$$x(t) = x(0) + t\dot{x}(0) + \frac{t^2}{2}\ddot{x}(0) - L^{-1}\dot{x}^3 - L^{-1}x^2\dot{x}, \tag{2.6}$$

$$x(t) = t - L^{-1}\dot{x}^3 - L^{-1}x^2\dot{x}. \tag{2.7}$$

Applying $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ to Eq.(2.7) we have

$$\sum_{n=0}^{\infty} x_n = t - L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}\sum_{n=0}^{\infty} B_n, \tag{2.8}$$

where

$$Nx = \dot{x}^3 = \sum_{n=0}^{\infty} A_n, \quad Rx = x^2\dot{x} = \sum_{n=0}^{\infty} B_n. \tag{2.9}$$

We find A_n and B_n from $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right]_{\lambda=0}, n = 0, 1, 2, 3, \ldots.$

$$\begin{cases} A_0 &= \dot{x_0}^3, \\ \\ A_1 &= x_1 N'(x_0) \\ &= \left(3\dot{x}_0^2\frac{d}{dt}\right)x_1 \\ &= 3\dot{x}_0^2\dot{x}_1, \\ \\ A_2 &= x_2N'(x_0) + \frac{1}{2}x_1^2N''(x_0) \\ &= \left(3\dot{x}_0^2\frac{d}{dt}\right)x_2 + \frac{1}{2}\left(3\dot{x}_0^2\frac{d}{dx_0dt} + 6\dot{x}_0\frac{d^2}{dt^2} + 6\dddot{x}_0\right)x_1^2 \\ &= 3\dot{x}_0^2\dot{x}_2 + \frac{1}{2}\left(12x_1\dot{x}_0\ddot{x}_1 + 12\dot{x}_0\dot{x}_1^2 + 6x_1^2\dddot{x}_0\right) \\ &= 3\dot{x}_0^2\dot{x}_2 + 6x_1\dot{x}_0\ddot{x}_1 + 6\dot{x}_0\dot{x}_1^2 + 3x_1^2\dddot{x}_0, \end{cases}$$

and

$$\begin{cases} B_0 &= x_0^2\dot{x}_0, \\ \\ B_1 &= x_1 R'(x_0) \\ &= \left(x_0^2\dfrac{d}{dt} + 2x_0\dot{x}_0\right)x_1 \\ &= x_0^2\dot{x}_1 + 2x_0\dot{x}_0 x_1, \\ \\ B_2 &= x_2 R'(x_0) + \dfrac{1}{2}x_1^2 R''(x_0) \\ &= \left(x_0^2\dfrac{d}{dt} + 2x_0\dot{x}_0\right)x_2 + \dfrac{1}{2}\left(x_0^2\dfrac{d}{dx_0 dt} + 4\dot{x}_0 + 2x_0\dfrac{d}{dt}\right)x_1^2 \\ &= x_0^2\dot{x}_2 + 2x_0\dot{x}_0 x_2 + \dfrac{1}{2}\left(4\dot{x}_0 x_1^2 + 4x_0 x_1\dot{x}_1\right) \\ &= x_0^2\dot{x}_2 + 2x_0\dot{x}_0 x_2 + 2\dot{x}_0 x_1^2 + 2x_0 x_1\dot{x}_1. \end{cases}$$

Returning A_n and B_n, we can determine the recursive relation which will be used to find the solution

$$\begin{aligned} x_0 =& t, \\ x_{n+1} =& -L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}\sum_{n=0}^{\infty} B_n. \end{aligned} \tag{2.10}$$

Solving this yields

$$\begin{cases} x_0 = & t, \\ x_1 = & -L^{-1}A_0 - L^{-1}B_0 \\ \quad = & -L^{-1}\dot{x}_0^3 - L^{-1}x_0^2\dot{x}_0 \\ \quad = & -\int\int\int dtdtdt - \int\int\int t^2 dtdtdt \\ \quad = & -\dfrac{1}{6}t^3 - \dfrac{1}{60}t^5, \\ \\ x_2 = & -L^{-1}A_1 - L^{-1}B_1 \\ \quad = & L^{-1}(3\dot{x}_0^2\dot{x}_1) - L^{-1}(x_0^2\dot{x}_1) - L^{-1}(2x_0\dot{x}_0 x_1) \\ \quad = & -3\left[\int\int\int(-\dfrac{1}{2}t^2 - \dfrac{1}{12}t^4)dtdtdt\right] - \left[\int\int\int t^2(-\dfrac{1}{2}t^2 - \dfrac{1}{12}t^4)dtdtdt\right] \\ & -2\left[t(-\dfrac{1}{6}t^3 - \dfrac{1}{60}t^5)dtdtdt\right] \\ \quad = & \dfrac{1}{40}t^5 + \dfrac{13}{2520}t^7 + \dfrac{1}{4320}t^9, \\ \\ x_3 = & -L^{-1}A_2 - L^{-1}B_2 \\ \quad = & -L^{-1}(3\dot{x}_0^2\dot{x}_2) - L^{-1}(6x_1\dot{x}_0\ddot{x}_1) - L^{-1}(6\dot{x}_0\dot{x}_1^2) - L^{-1}(3x_1^2\dddot{x}_0) - L^{-1}(x_0^2\dot{x}_2) \\ & -L^{-1}(2x_0\dot{x}_0 x_2) - L^{-1}(2\dot{x}_0 x_1^2) - L^{-1}(2x_0 x_1\dot{x}_1) \\ \quad = & -\dfrac{23}{1680}t^7 - \dfrac{261072}{91445760}t^9 - \dots. \end{cases}$$

Consequently, the solution in a series form is expressed by

$$\begin{aligned} x(t) = & \ t - \frac{1}{6}t^3 - \frac{1}{60}t^5 + \frac{1}{40}t^5 + \frac{13}{2520}t^7 - \frac{23}{1680}t^7 + \frac{1}{4320}t^9 \\ & - \frac{261072}{91445760}t^9 + \ldots \\ = & \ t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{362880}t^9 - \ldots, \end{aligned} \tag{2.11}$$

which is the Taylor series to the exact solution

$$x(t) = \sin t. \tag{2.12}$$

Consequently, the solution to Eq.(2.1), with respect to IC determined in Eq.(1.4), is

$$\begin{aligned} x(t) &= \frac{B}{\omega}\sin\omega t, \qquad x(0) = 0, \\ \dot{x}(t) &= B\cos\omega t, \qquad \dot{x}(0) = B, \\ \ddot{x}(t) &= -\omega B\sin\omega t, \quad \ddot{x}(0) = 0, \\ \dddot{x}(t) &= -\omega^2 B\cos\omega t, \end{aligned} \tag{2.13}$$

where ω is the angular frequency. Then the period T is defined as

$$T = \frac{2\pi}{\omega}, \tag{2.14}$$

and the approximate displacement amplitude A is given by

$$A = \frac{B}{\omega}, \tag{2.15}$$

where $B = \dot{x}(0)$ is the initial velocity amplitude.

Substituting Eq.(2.13) into Eq.(2.1), and manipulating using trigonometric identities, yields, for $B \neq 0$,

$$\begin{aligned} -\omega^2 B\cos\omega t &= -B^2\cos^3\omega t - \frac{B^3}{\omega^2}\sin^2\omega t\cos\omega t, \\ \omega^2\cos\omega t &= B^2\cos^3\omega t + \frac{B^2}{\omega^2}\Big(\cos\omega t - \cos^3\omega t\Big), \\ \omega^2\cos\omega t &= \frac{B^2}{4}\Big(4\cos^3\omega t\Big) + \frac{B^2}{4\omega^2}\Big(4\cos\omega t - 4\cos^3\omega t\Big), \\ \Longrightarrow \omega^2\cos\omega t &= \frac{B^2}{4\omega^2}\Big(3\omega^2 + 1\Big)\cos\omega t + \frac{B^2}{4\omega^2}\Big(\omega^2 - 1\Big)\cos 3\omega t. \end{aligned} \tag{2.16}$$

Comparing the coefficients of like terms, we get

$$\omega^2 = \frac{B^2}{4\omega^2}\Big(3\omega^2 + 1\Big). \tag{2.17}$$

This equation can be represented as

$$\omega^4 - \frac{3B^2}{4}\omega^2 - \frac{B^2}{4} = 0. \tag{2.18}$$

Solving for positive ω^2 gives

$$\omega = \frac{1}{2\sqrt{2}}\sqrt{3B^2 + \sqrt{9B^4 + 16B^2}}. \tag{2.19}$$

The numerical values for T in Eq.(2.14), resulting from the approximate expression in Eq.(2.19), for a range of values of B, are compared with the exact values attained by solving the 3^{rd} ODE Eq.(2.1) with initial conditions given by Eq.(2.13) in Table 1.

Table 1. Comparison of the approximate periods of ADM and Perturbation Method with the exact periods

B	T(AD method)	T(Perturbation Method) [1]	T(exact) [1]
0.1	7.065998	27.065998	25.359725
0.2	18.438632	18.438632	17.495410
0.5	10.461083	10.461083	10.210761
1	6.283185	6.283185	6.283185
2	3.457326	3.457326	3.508793
5	1.438527	1.438527	1.468638
10	0.723920	0.723920	0.739762
20	0.362559	0.362559	0.370580

The accuracy of Adomian Decomposition Method (ADM) deduced from Table 1 is very good throughout the wide range of B.

Near $B = 1$, the accuracy is the best and when B departs from this value, it worsens slightly. If $B = 1$, then $\omega = 1$ and $x(t) = \sin t$, which is completely in agreement with an exact solution (ES) of Eq.(2.1) as illustrated by the figures.

The figures from 1 to 8 display the approximate periodic solution (APS) investigated by LDM compared with the exact one for different values of B. Also, the ADM method proves identical results to those obtained by perturbation method.

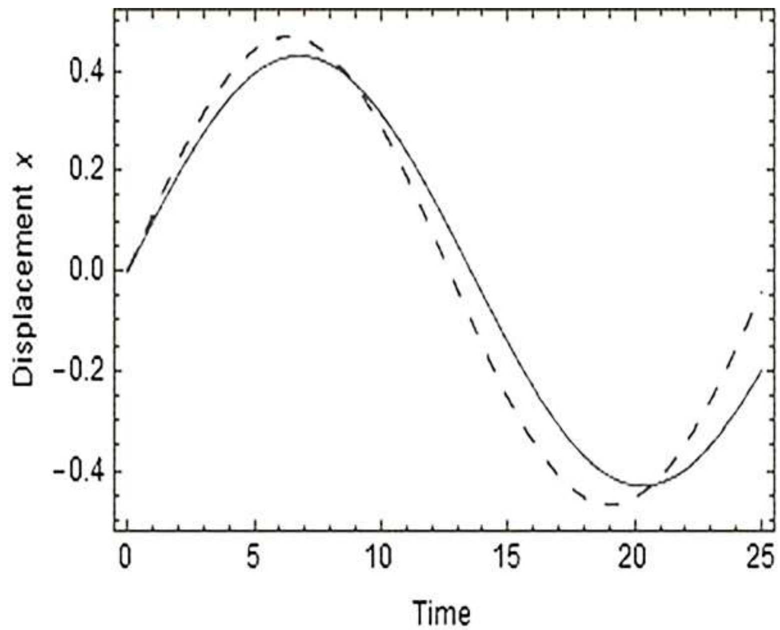

Figure 1. Comparison of the APS x using ADM (dashed line) with the ES x_e (solid line) for $B = 0.1$.

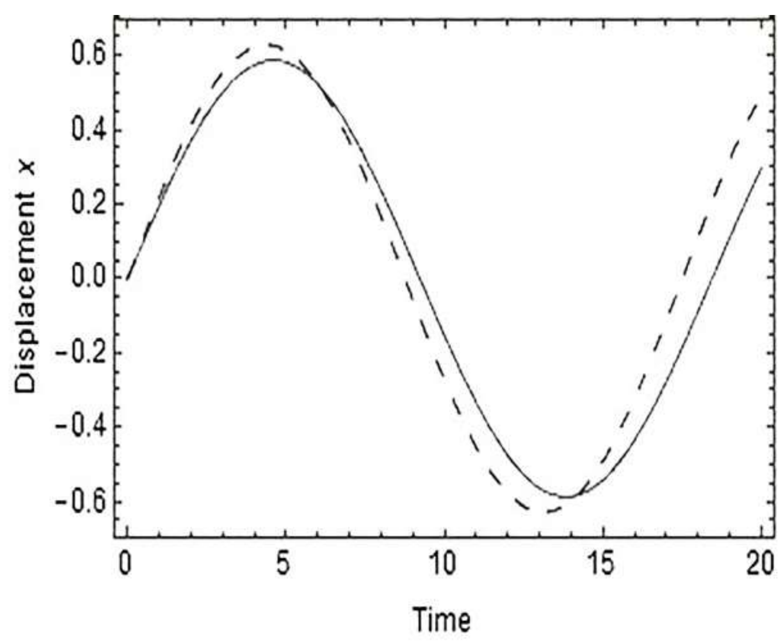

Figure 2. Comparison of the APS x using ADM (dashed line) with the ES x_e (solid line) for $B = 0.2$.

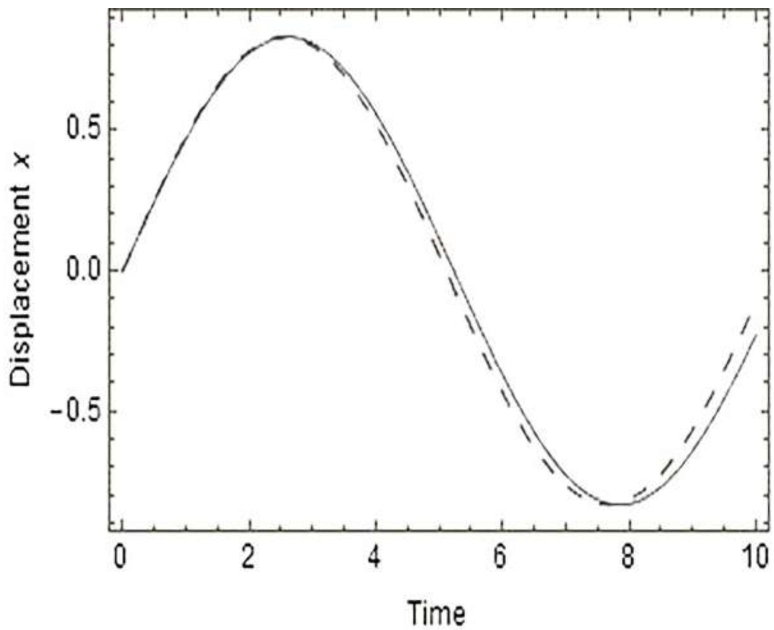

Figure 3. Comparison of the APS x using ADM (dashed line) with the ES x_e (solid line) for $B = 0.5$.

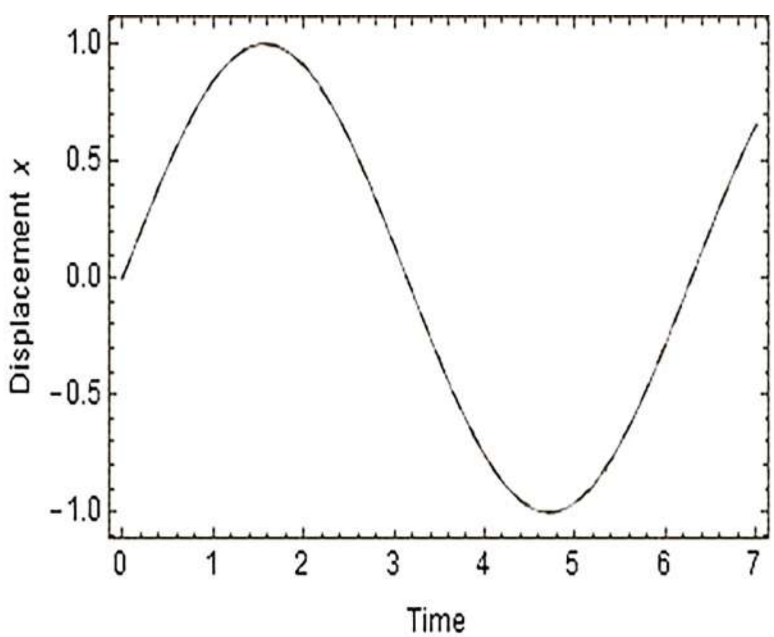

Figure 4. Comparison of the APS x using ADM (dashed line) with the ES x_e (solid line) for $B = 1$.

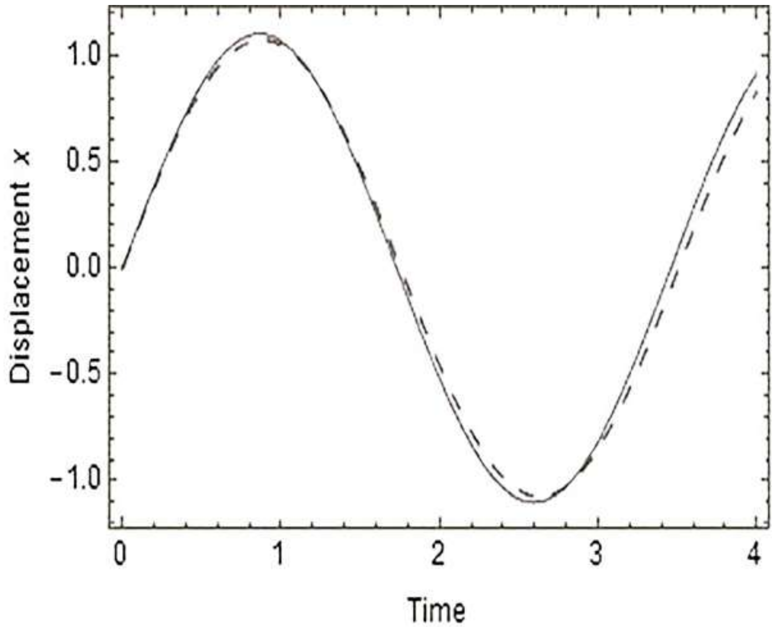

Figure 5. Comparison of the APS x using ADM (dashed line) with the ES x_e (solid line) for $B = 2$.

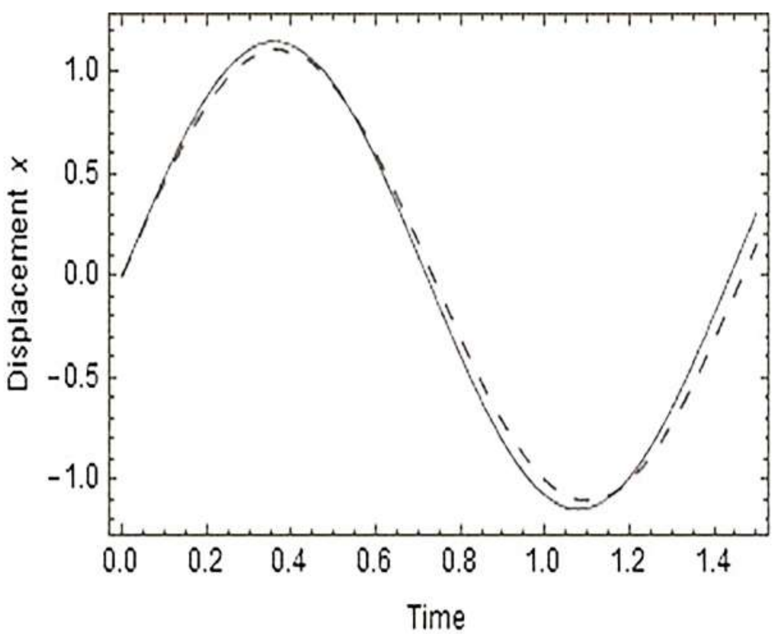

Figure 6. Comparison of the APS x using ADM (dashed line) with the ES x_e (solid line) for $B = 5$.

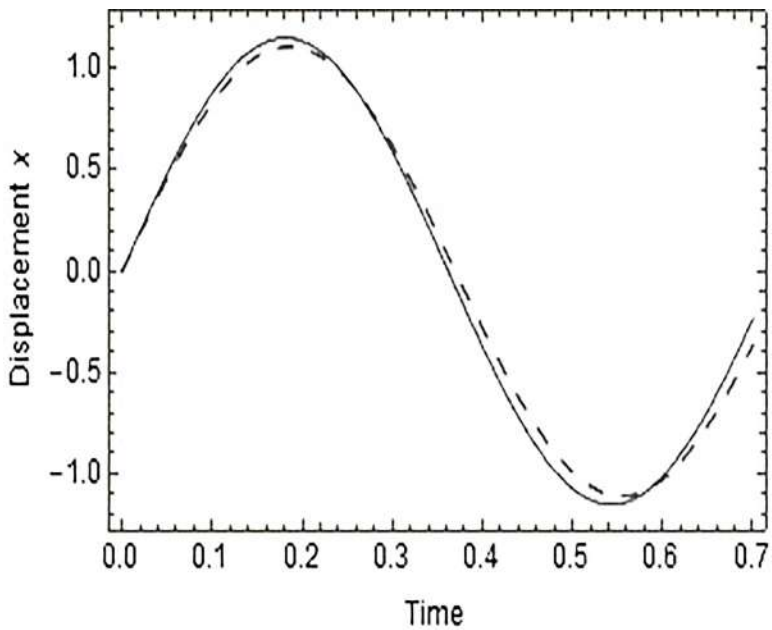

Figure 7. Comparison of the APS x using ADM (dashed line) with the ES x_e (solid line) for $B = 10$.

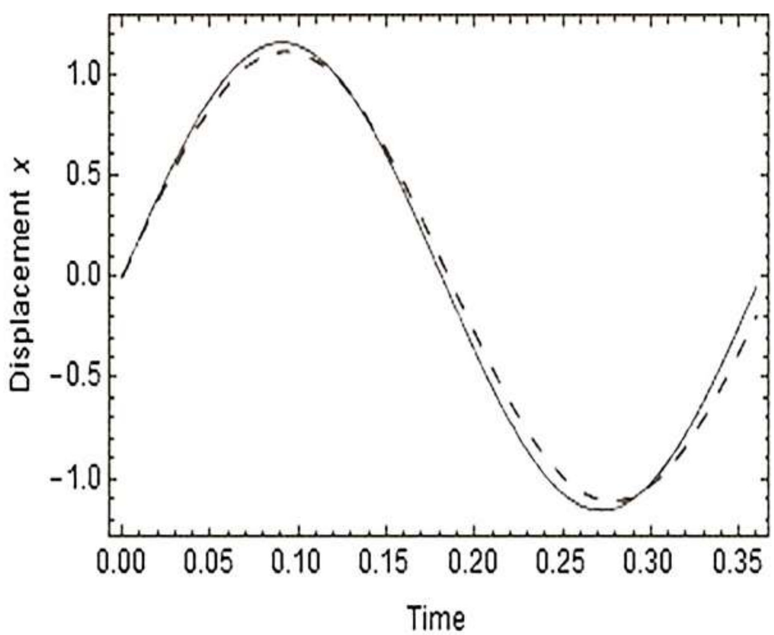

Figure 8. Comparison of the approximate periodic solution x using ADM (dashed line) with the ES x_e (solid line) for $B = 20$.

3. Conclusion

LDM succeeded highly in demonstrating the JE and proved a great consistency with the ES. The approximate solutions of the JE are demonstrated by figures and show convergence with the ES. The convergence of the approximate solutions of the JE increases when B approaches 1 and is exact when B = 1. LDM proves highly its advantage in demonstrating an exact or approximate solution for nonlinear phenomena such as the nonlinear JE.

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