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Critical Point Theorems of Non-smooth Functionals without the Palais-Smale Condition

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Abstract This paper introduces some new variants of abstract critical point theorems that do not rely on any compactness condition of Palais Smale type. The focus is on locally Lipschitz continuous functional $\Phi : E \to R$, where E is a reflexive banach space. The theorems are established through the utilization of the least action principle, the perturbation argument, the reduction method, and the properties of sub-differential and generalized gradients in the sense of F.H. Clarke. These approaches have been instrumental in advancing the theory of critical points, providing a new perspective that eliminates the need for traditional compactness constraints. The implications of these results are far-reaching, with potential applications in optimization, control theory, and partial differential equations.

Keywords Critical point, minimax theorems, locally Lipschitz functional, the least action principle, perturbation argument

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1. Introduction

Critical point theory has long been a cornerstone in the study of the existence and multiplicity of solutions for various classes of nonlinear problems. Traditionally, this domain has relied heavily on specific compactness assumptions, notably the Palais-Smale condition [3,15,23]. Notwithstanding, these presumptions are not universally applicable, particularly concerning certain substantial classes of functionals. This underscores the importance of developing innovative critical point theorems that are free from these constraints for broader applicability [2,10,16,21,22,25].

The Palais-Smale condition is a classical compactness assumption that is often used to prove the existence of critical points for functionals on Banach spaces. However, this condition may not hold for some important classes of functionals, such as non-smooth or non-convex ones. Therefore, many researchers have tried to develop new critical point theorems that can overcome this limitation, see for example [6,9], or apply to more general settings, see for example [1,5,19,26].

However, the Palais-Smale condition proved to be too restrictive for many modern variational problems that involve more complicated functionals. In response, researchers looked for more flexible frameworks that could handle non-compact situations, leading to the emergence of critical point theories that do not depend on

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this traditional compactness constraint.

The study in question has its roots to the early and mid-20th century, during which mathematicians like Marston Morse [17] and R. Palais [20] established the fundamental principles of critical point theory by building upon the variational methods developed by Euler and Lagrange. Morse theory, for instance, unveiled the correlation between the topology of manifolds and the critical points of smooth functions.

This study focuses on the category of locally Lipschitz continuous functionals $\Phi : E \to \mathbb{R}$, where E is a reflexive Banach space. We present novel variations of abstract critical point theorems that eliminate the need for any compactness criterion of Palais-Smale condition. The primary methodologies we employ include the least action principle, the perturbation argument, the reduction approach, and the characteristics of sub-differential and generalized gradients as defined by F.H. Clarke [11]. These strategies enable us to acquire critical points for functionals that may exhibit non-smooth or non-convex components, and to handle different sorts of restrictions and boundary conditions. By eliminating the need for traditional compactness constraints, these new critical point theorems offer far-reaching implications, with potential applications in optimization, control theory, and partial differential equations. The generalizations presented in this paper provide a new perspective on critical point theorems, making them applicable to a wider range of functionals and settings, and eliminating the need for traditional compactness constraints.

A.C. Lazer, E.M. Landesman and D.R. Meyers in [13, Theorem 1] established the existence and uniqueness of critical points for certain functional without the compactness conditions. This theorem provided conditions under which a real valued function defined on a real Hilbert space has a unique minimizer. The theorem uses a variational argument and the saddle point principle to demonstrate the existence and uniqueness of the critical point.

Bates and Ekeland [4], Manasevich [14] and A.C. Lazer [13] generalized [13, Theorem 1] to the case where X and Y are not necessarily finite dimensional or by weakening the conditions:

$$(D^2\Phi(u)h, h) \leq -m_1\|h\|^2, \tag{1.1}$$

$$(D^2\Phi(u)k, k) \geq m_2\|k\|^2 \tag{1.2}$$

for all $u \in H, h \in X$ and $k \in Y$.

Moussaoui and Gossez [18] generalized Theorem 1 in [13] by relaxing conditions (1.1) and (1.2) to conditions of coercivity and concavity and they supposed that Φ is of class $\mathcal{C}^1$ instead of $\mathcal{C}^2$. They proved the following theorem:

Theorem 1.1. *Let H be a Hilbert space such that : $H = V \oplus W$, where V is finite dimensional subspace of H and W its orthogonal space. Consider a functional $\Phi : H \to \mathbb{R}$ that satisfies the following conditions:*
(i) Φ is of class $\mathcal{C}^1$.
(ii) Φ is coercive on W.
(iii) For a fixed $w \in W$, the mapping $v \mapsto \Phi(v + w)$ is concave on V.
(iv) For a fixed $w \in W$, $\Phi(v + w) \to -\infty$ when $\|v\| \to +\infty$, $v \in V$; and this