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Shallow-Water Models with the Weak Coriolis and Underlying Shear Flow Effects*

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Abstract In this paper, we are committed to deriving shallow-water model equations from the governing equations in the two-dimensional incompressible fluid with the effects of weak Coriolis force and underlying shear flow. These approximate models are established by working within a weakly nonlinear regime, introducing suitable far-field or near-field variables, and truncating the asymptotic expansions of the unknowns to an appropriate order. The obtained models generalize the classical KdV and Boussinesq equations, as well as KdV and Boussinesq equations with the Coriolis or shear flow effects.

Keywords KdV equation, Boussinesq equation, Coriolis effect, shear flow

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1. Introduction

The study of geophysical water waves is a fascinating subject in recent years. Within a certain large-scale geophysical water waves, fluid dynamics is mainly affected by the interaction between gravity and Coriolis force generated by the Earth's rotation. There are various shallow-water models with the Coriolis effect proposed as approximations to the governing equations for gravity water waves under different nonlinear regimes. In the Boussinesq scaling (weakly nonlinear regime), one can derive the geophysical Kortewe-de Vries (gKdV) equation [12]

$$2\eta_t - 2\omega_0\eta_x + 3\eta\eta_x + \frac{1}{3}\eta_{xxx} = 0,$$

where η is relevant to the free surface elevation and ω_0 is a constant related to the Coriolis effect. Recently, a modified version of the geophysical KdV of the above equation has been established in [1], which is called gpKdV equation. Additionally, the geophysical Boussinesq-type (gBouss) equation is also obtained in [12]

$$H_{tt} - 2\omega H_{tX} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} = 0,$$

where H denotes the free surface elevation and ω is a constant related to the Coriolis effect. The new features of the Coriolis term introduced into a geophysical Boussinesq system in the long-wave assumption is considered in [14]. In the Camassa-Holm

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(CH) scaling (moderately nonlinear regime), the rotation-CH (R-CH) equation with the Coriolis effect is derived from incompressible and irrotational two-dimensional equatorial shallow water [13]. Recently, a highly nonlinear shallow-water model with the Coriolis effect has been proposed under a larger scaling than the CH one [19].

On the other hand, in order to represent another aspect of realistic observed flows, it requires us to study the propagation of various types of weakly nonlinear long waves in a flow moving according to some prescribed vorticity. Such a flow is often called a shear flow [8,18]. Due to the computational complexities for the case of an arbitrary shear flow, for simplicity (of course, non-trivial), most of the shallow-water models are derived with an underlying linear shear flow. It is known that the linear shear flow means constant vorticity. Recently, two-dimensional water-wave problem with a general non-zero vorticity field in a fluid volume with a flat bed and a free surface has been studied in [15]. The CH equation is relevant to water waves moving over a linear shear flow established by Johnson [18] via a double asymptotic expansion. In [21], a generalized CH equation in the shallow-water regime under the CH scaling with a linear shear is derived. Additionally, the Boussinesq equation with constant vorticity is given in [8]. For the case of general shear flow, an example is the KdV equation satisfied by the free surface elevation proposed in [10,18].

In this manuscript, we first derive a generalized KdV equation satisfied by the horizontal component of the velocity field with the effects of weak Coriolis and an arbitrary shear flow. It reads,

$$u_t + c_1 u_x + \alpha\varepsilon u u_x + \beta\mu u_{xxx} = 0, \tag{1.1}$$

where u is the fluid velocity in the horizontal direction. The coefficients in Eq. (1.1) depend on the right-going wave speed c, the constant rotational frequency Ω caused by the Coriolis effect, and the arbitrary underlying shear flow $U(z)$, given by

$$c_1 = c + \frac{c\Omega\gamma_1}{I_{31}}, \qquad \alpha = -\frac{3I_{41}\gamma_1}{2I_{31}}, \qquad \beta = -\frac{J_1}{2I_{31}},$$

where $\gamma_1 = -\frac{1}{((U(z)-c)I_2)'}$, $I_2 = \int_0^z \frac{1}{(U(z)-c)^2}\mathrm{d}z$, $I_{31} = \int_0^1 \frac{1}{(U(z)-c)^3}\mathrm{d}z$, $I_{41} = \int_0^1 \frac{1}{(U(z)-c)^4}\mathrm{d}z$, and $J_1 = \int_0^1\int_Z^1\int_0^\zeta \frac{(U(\zeta)-c)^2}{(U(Z)-c)^2(U(z)-c)^2}\mathrm{d}z\mathrm{d}\zeta\mathrm{d}Z$.

Particularly, in absence of the Coriolis effect and any shear flow ($\Omega \equiv 0$, $U(z) \equiv 0$), we can get the classical KdV equation from Eq. (1.1). When $U(z) \equiv 0$, in contrast to the gKdV equation of the free surface elevation, we obtain a KdV equation that the horizontal velocity satisfies only with the Coriolis effect. The reason why our equation is slightly different from gKdV is that we here use the far-field variables $\xi = \sqrt{\varepsilon}(x - ct), \tau = \varepsilon\sqrt{\varepsilon}t$, not $\xi = x - ct, \tau = \varepsilon t$. Moreover, as mentioned in Remark 2.1 of [13], it enables us to derive another KdV equation only with the Coriolis effect of the horizontal velocity under the Boussinesq scaling in the form

$$u_t + cu_x + 3\nu\varepsilon u u_x + \frac{\nu}{3}c\mu u_{xxx} = 0, \tag{1.2}$$

where $\nu = \frac{c^2}{c^2+1}, c = \sqrt{1+\Omega^2} - \Omega$ and Ω is the constant rotational frequency due to the Coriolis effect. Note that our equation is also different from Eq. (1.2), because we here use the weak Coriolis effect, not Coriolis effect, which leads to the difference in equations at the leading-order approximation. And we thus obtain $c = 1$ from the Burns condition when $U(z) \equiv 0$. This is really different from c appearing in Eq.