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Existence and Uniqueness of Solutions of Nonlinear Integral Equations through Results in Fuzzy Bipolar Metric Spaces

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Abstract In this paper the concept of fuzzy bipolar metric spaces are investigated and their properties in relation with fixed points by the consideration of triangular property of fuzzy bipolar metric are explored. The study presents a series of established results under covariant and contravariant mappings supported by illustrative examples and discusses the implications of these findings. Furthermore, the established results are applied to demonstrate the existence and uniqueness of solutions to nonlinear integral equations. The exploration of fuzzy bipolar metric spaces and their application to integral equations provides valuable insights into the field of mathematical analysis and opens avenues for further research.

Keywords Triangular property, fuzzy metric space, fuzzy bipolar metric space, covariant mappings, contravariant mappings

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1. Introduction

Fixed point theory has become a central focus across various disciplines, such as engineering, optimization, physics, economics, and mathematics. The foundation of this theory was significantly bolstered by the Banach fixed point theorem, introduced by Banach [2] which has catalyzed extensive research in both the mathematical and scientific spheres.

In 1975, Kramosil and Michalek [12] introduced a groundbreaking concept of fuzzy metric spaces. This innovation was built upon the groundwork of introduction of the continuous t-norm laid by Schweizer and Sklar [22] in 1960. The pivotal role played by L.A. Zadeh's fuzzy set theory, formulated in 1965 [28], cannot be overstated in this context. Subsequently, George and Veeramani [8] further extended the framework of fuzzy metric spaces by incorporating the Hausdorff topology and adapting established theorems from classical metric spaces. This extension yielded profound results in fuzzy metric space and its generalisations [1,5,9–11,16,20,23–27].

In a more recent mathematical advancement, Mutlu and Gurdal [17] introduced bipolar metric spaces in 2016. Unlike conventional metric spaces, which exclusively explore distances within a single set, bipolar metric spaces allow for the examination of distances between points drawn from two distinct sets. Following this

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development, subsequent researchers [6, 7, 13, 18, 19] delved into the realm of fixed point theorems within bipolar metric spaces, uncovering diverse applications across various contexts. Building upon this foundation, Bartwal et al. [3] introduced the notion of fuzzy bipolar metric spaces, extending the concept of fuzzy metric spaces to this new context by introducing a novel distance measurement scheme for points residing in different sets. This extension paved the way for the derivation of significant fixed point results within fuzzy bipolar metric spaces [4, 14, 15, 21]. The research gap addressed in our work lies in the establishment of new fixed point results in fuzzy bipolar metric spaces, specifically focusing on the generealising the constraint satisfied by self mappings by introducing the concept of control function, and taking under consideration the triangular property of induced fuzzy bipolar metric. While existing literature provides valuable insights into fixed point theory and fuzzy bipolar metric spaces, there is a need to exapand the implications of considering control function in fuzzy bipolar metric spaces. By bridging this research gap, our paper contributes to the theoretical development of generalisation of fuzzy metric spaces and expands the understanding of fixed point theory. With the consideration of control function and self-mappings with triangular property, the existing framework offers a versatile foundation that can be applied across various domains.

In this study, we embark on an exploration of the fundamental constructs of fuzzy bipolar metric spaces, introduced in Section 2. Subsequently, in Section 3, we establish some fixed point results within fuzzy bipolar metric spaces, incorporating the distinctive triangular property inherent in fuzzy bipolar metrics and implimenting the use of control function. Section 4 provides illuminating examples that provide empirical support for the established fixed point results, and Section 5 engages in a comprehensive discussion of the broader implications arising from these findings. Further, in Section 6, we showcase the practical application of these results by demonstrating their utility in establishing the existence and uniqueness of solutions for nonlinear integral equations. Ultimately, in Section 7, we summerize the findings and discussion in conclusion section.

2. Preliminaries

In this section, we provide some fundamental definitions and properties to establish the main results of this article for compact maps.

Definition 2.1. [22] Suppose there is a binary operation $*$ from $[0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

a) Commutativity and associativity of $*$;
b) Continuity of $*$;
c) $k * 1 = k$, for all $k \in [0,1]$;
d) $j * l \leqslant o * q$ whenever $j \leqslant o$, $l \leqslant q$ and $j, l, o, q \in [0,1]$.

Definition 2.2. [8] Consider a non-empty set M. Let $*$ be a t-norm which is continuous and S be a fuzzy set on $M \times M \times (0, \infty)$. Then, $(M, S, *)$ is called a fuzzy metric space if for all $m, i, j \in M$ and $r, s > 0$, the following conditions hold:

i) $S(m, i, r) > 0$;
ii) $S(m, i, r) = 1$ if and only if $m = i$;
iii) $S(m, i, r) * (i, j, s) \leqslant S(m, j, r + s)$;
v) $S(m, i, \_) : (0, \infty) \to (0, 1]$ is continuous.