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# Dynamics of an $n$-Patch Predator-Prey Model with Allee Effect*

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**Abstract** A class of $n$-patch predator-prey diffusion models with the Allee effect is established. The influence of the Allee effect and diffusion of prey on the existence and stability of the equilibrium point are investigated. Firstly, sufficient conditions for the permanence and extinction of the system are analyzed. Secondly, by constructing a new Lyapunov function in terms of graph theory, we obtain a sufficient condition of the global asymptotical stability for the positive equilibrium point. Finally, our results of this paper are verified by Matlab simulation.

**Keywords** $n$-patch, stability, Allee effect, predation-prey model

**MSC(2010)** 34A12, 34D23.

## 1. Introduction

Considering the influence of predator and prey diffusion behavior, Levin [1] first established the population dynamics model in a patch environment, and studied the related problems of predator-prey diffusion model [2–9]. Up to now, the dynamics problem of patch predator-prey model has received lots of attention. For example, Kuang and Takeuchi [2] investigated the following autonomous predator-prey systems

$$\begin{aligned}\dot{x}_1 &= x_1 g_1(x_1) - y p_1(x_1) + d(x_2 - x_1),\\ \dot{x}_2 &= x_2 g_2(x_2) - y p_2(x_2) + d(x_1 - x_2),\\ \dot{y} &= y[-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2)].\end{aligned}$$

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They obtained sufficient conditions for permanence, local stability and global stability of the system. After this, many scholars have extended Kuang and Takeuchi's model to consider the influence of impulse [7], fear effect [8], age structure [9] and other factors on the patch predator-prey model. Since the dynamic analysis in [2–9] is mainly concentrated in the patch environment, the patch predator-prey model for high-dimensional systems is not applicable. Li and Shuai [10] considered the $n$-patch predator-prey model. In [10], the authors used the results of graph theory to construct the global Lyapunov function of a large-scale coupled system from a single vertex system, and then obtained the stability of the system.

In addition, population density is limited by the environment. Excessive sparseness or overcrowding can inhibit the growth of the population, which means that the species has the range for the population growth. This phenomenon is called the Allee effect [11–14]. In this paper, we assume that the prey growth rate is affected by the Allee effect, so in the absence of predators, the per capita growth rate function becomes $g(x) = r\left(1 - \frac{x}{k}\right)(x + m)$, where $m$ is the Allee effect parameter. Recently, Pal, Samanta [15] and Saha [16] studied the diffusion dynamics of a predator-prey system with a strong Allee effect in a two-patch environment. In [15, 16], they obtained the existence and stability criteria of the positive equilibrium point in the presence and absence of diffusion. Biswas and Pal [17] introduced the strong Allee effect into the three-patch model. Biswas and Pal showed the dynamics and the asymptotic behavior of the system and proved the occurrence of Hopf bifurcation. The Allee effect plays an important role in the dynamic behavior of the predator prey system. However, there are few results to deal with the weak Allee effect of $n$-path predator-prey.

Motivated by the above discussions, we discuss the weak Allee effect of $n$-path predator-prey in this paper. By using Li and Shuai's technique of constructing Lyapunov function, the conditions for the stability of the positive equilibrium point of the $n$-patch predator-prey model are given. We also provide examples to demonstrate the effectiveness of the proposed stability results. It also shows how the weak Allee effect determines the existence and stability of positive equilibrium.

This paper is organized as follows: In Section 2, our mathematical model of $n$-patch predator-prey with Allee effect is presented and some preliminaries are given. In Section 3, the main results for both permanence and asymptotically stability of $n$-patch predator-prey model with the weak Allee effect are proposed. An effective numerical simulation is presented in Section 4 to illustrate the main results. Concluding remarks are collected in Section 5.

## 2. The model and preliminaries

In this paper, we consider the following system

$$\begin{cases} \frac{dx_i}{dt} = r_i(1 - \frac{x_i}{k_i})(x_i + m_i)x_i - b_i {x_i}^2 - q_i x_i y + \sum\limits_{l=1}^{n} d_{il}(x_l - x_i), \\ \frac{dx_n}{dt} = r_n(1 - \frac{x_n}{k_n})x_n + \sum\limits_{l=1}^{n} d_{nl}(x_l - x_n), \\ \frac{dy}{dt} = -sy + y\sum\limits_{i=1}^{n-1} e_i x_i . i = 1, 2, ..., n-1. \end{cases} \tag{2.1}$$

In the system, $n \geq 2$, $x_l(t)$ is the number of prey populations in the $l, l = 1, 2, ..., n$,