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On a Class of Discrete Problems with the $p(k)$-Laplacian-like Operators

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Abstract In this paper, we consider a nonlinear discrete problem originating from a capillary phenomena, involving the $p(k)$-Laplacian-like operators with mixed boundary condition. Under appropriate assumptions on the function f and its primitive F near zero and infinity, we investigate the existence and multiplicity of nontrivial solutions by using variational methods and critical point theory.

Keywords Critical point theory, discrete problems, variational methods, $p(k)$-Laplacian-like operators

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1. Introduction

The aim of this article is to establish the existence and multiplicity of solutions for a general discrete problem originating from capillary phenomena, involving the $p(k)$-Laplacian-like operators with mixed boundary condition of the following form:

$$(P)\quad \begin{cases} -\Delta\big(m(r-1)a(r,\Delta u(r-1))\big)+\beta(r)|u(r)|^{p(r)-2}u(r)=f(r,u(r)),\ r\in[1,N]_{\mathbb{N}},\\ u(0)=\Delta u(N)=0, \end{cases}$$

where $a(.,.)$ is defined as follows:

$$a(r,s)=\Big(1+\phi_c\left(|s|^{p(r-1)}\right)\Big)|s|^{p(r-1)-2}s,\ \text{ for all } r\in[1,N]_{\mathbb{N}} \text{ and } s\in\mathbb{R},$$

and ϕ_c is the so-called mean curvature operator defined as ([23])

$$\phi_c(s):=\frac{s}{\sqrt{1+s^2}},\ s\in\mathbb{R}.$$

Let $[1,N]_{\mathbb{N}}$ be the discrete interval given by $[1,N]_{\mathbb{N}}:=\{1,2,\dots,N\}$, where $N\geq 2$ is a positive integer and Δ denotes the forward difference operator $\Delta u(r):=u(r+1)-u(r)$. In addition, $m:[0,N+1]_{\mathbb{N}}\longrightarrow[1,+\infty)$, $\beta:[0,N+1]_{\mathbb{N}}\longrightarrow[1,+\infty)$

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and $p : [0, N+1]_{\mathbb{N}} \to [2, +\infty)$ are given functions, and for every fixed $r \in [0, N]_{\mathbb{N}}$, $f(r, .) : \mathbb{R} \to \mathbb{R}$ is a continuous function that checks some conditions mentioned below. We say that a function $u : [0, N+1]_{\mathbb{N}} \to \mathbb{R}$ is a solution of problem (P) if it satisfies both equations of (P).

For convenience, for any bounded function $h : [0, N+1]_{\mathbb{N}} \longrightarrow \mathbb{R}$, we will use the following symbols

$$h^+ := \max_{r \in [0,N+1]_{\mathbb{N}}} h(r), \quad h^- := \min_{r \in [0,N+1]_{\mathbb{N}}} h(r).$$

Let

$$\begin{aligned} F(t,s) &:= \int_0^s f(t,\zeta)d\zeta, \\ A(t,s) &:= \int_0^s a(t,\zeta)d\zeta \\ &= \frac{1}{p(t-1)}\left(|s|^{p(t-1)} + \sqrt{1+|s|^{2p(t-1)}} - 1\right), \end{aligned}$$

for all $(t, s) \in [0, N]_{\mathbb{N}} \times \mathbb{R}$, and we put

$$C_{m,\beta} = (m^+ 2^{p^-} + \beta^+) \geq 1, \quad K_0 = \left\{(2N+2)\max\left\{m^+, \beta^+\right\}\right\}^{\frac{p^- - p^+}{p^+ p^-}} \leq 1. \quad (1.1)$$

Now, we introduce the following assumptions on the nonlinear term f and its primitive F at zero and infinity:

(H_1) There exists $\eta < \frac{1}{p^+ C_{m,\beta}}$ such that for all $r \in [1, N]_{\mathbb{N}}$,

$$\limsup_{|x| \to \infty} \frac{F(r,x)}{|x|^{p^-}} \leq \eta.$$

(H_2) There exists $\delta > 0$ such that for all $r \in [1, N]_{\mathbb{N}}$,

$$B_0(r) := \liminf_{x \to 0} \frac{F(r,x)}{|x|^{p^-}} \geq \delta.$$

(H_3) $f(r, -x) = -f(r, x)$ for all $(r, x) \in [1, N]_{\mathbb{N}} \times \mathbb{R}$, i.e., $f(r, x)$ is odd in x.

(H_4) $\lim_{x \to 0} \frac{F(r,x)}{|x|^{p^+}} = 0$, for all $r \in [1, N]_{\mathbb{N}}$.

(H_5) There exists κ such that $\kappa > \frac{2}{p^-} N^{\frac{p^+ - p^-}{p^-}} K_0^{p^+} C_{m,\beta}^{\frac{p^+}{P^-}}$ and

$$\liminf_{|x| \to \infty} \frac{F(r,x)}{|x|^{p^+}} \geq \kappa, \text{ for all } r \in [1, N]_{\mathbb{N}}.$$

In recent years, there have been more and more papers on the topic of difference equations with a variable exponent, which have contributed to the development of research and studies related to the problems of differential equations since it has been frequently used as mathematical models in several domains [9, 14]. Known means from the critical point theory are applied to prove the existence of solutions.