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Global Stability of a Parabolic System Involving a Nonlinear Diffusion Operator with an Example in Epidemiology

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Abstract The purpose of this work is to study the global stability of specific nonlinear parabolic equations incorporating the p-Laplacian operator, with a primary emphasis on their application in biology, particularly in the context of epidemiology. This investigation entails the construction of Lyapunov functions derived from the associated ODEs. To elucidate our approach, we provide an illustrative example from the field of epidemiology.

Keywords Epidemiological model, Lyapunov function, parabolic systems, p-Laplacian operator

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1. Introduction

Mathematical models have long been indispensable tools in understanding and predicting the dynamics of infectious diseases. By quantifying the complex interactions between pathogens, hosts, and populations, these models enable researchers and policymakers to gain insights into the spread of diseases and evaluate potential control strategies. One of the pioneering works in epidemic modeling can be attributed to Kermack and McKendrick [2], who introduced the influential SIR model in 1927. Over time, epidemic mathematical models have evolved and expanded to incorporate additional complexities. Researchers recognized the importance of accounting for factors such as age structure, spatial heterogeneity, and varying transmission rates. This led to the development of more sophisticated compartmental models, such as the SEIR model that introduced an exposed compartment [9]. Moreover, spatial epidemic models and network-based models emerged to capture the influence of geographical locations and social connections on disease spread [1]. In recent years, advanced mathematical techniques have further enhanced the capabilities of epidemic models.

In recent studies, a multitude of researchers in the fields of epidemiology and virology have employed spatiotemporal equations. The research [3], for instance, analyzed an epidemic model described as a parabolic system of PDEs, incorporating the p-Laplacian operator. Their primary objective was to devise a well-thought-out optimal control approach within a specified spatiotemporal context. This approach

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was meticulously crafted to mitigate both the propagation of infections and the associated vaccination expenses. Through a Lyapunov function, [4] demonstrated the global asymptotic stability of the endemic equilibrium in a graph Laplacian reaction-diffusion SIR system when the basic reproduction number $\mathcal{R}_0$ is greater than 1. Conversely, they also established the global asymptotic stability of the disease-free equilibrium when the basic reproduction number is less than 1.

An independent investigation was carried out by [7], where an Ebola epidemic model was developed, taking into account constraints posed by limited medical resources, immunity loss, and the implementation of measures such as tracking and quarantining susceptible individuals. Meanwhile, the stability analysis of the disease-free equilibrium is presented, and the existence of multiple endemic equilibria as well as the occurrence of bifurcation are deduced. Utilizing the next-generation matrix method, the study computed the basic reproduction number $\mathcal{R}_0$ and subsequently analyzed the model's stability. A separate study by [10] explored an SIRS epidemic model incorporating logistic growth and information intervention. The study introduced the basic reproduction number $\mathcal{R}_0$ and presented key findings about local stability. Furthermore, the research derived sufficient conditions for the global stability of the endemic equilibrium. In the study by [6], an intracellular time delay was incorporated into an HBV model originally proposed in [5]. This time delay accounts for the lag between cell infection and the generation of new virus particles. The investigation was conducted within a one-dimensional interval with Neumann boundary conditions, with the simplifying assumption of homogeneous space to facilitate the establishment of global stability for equilibrium solutions. Furthermore, [8] extended the model introduced by [6] by introducing a saturation response mechanism and derived the necessary conditions for ensuring the global stability of the infected steady state.

The motivation for this paper lies in the profound significance of understanding and establishing the global stability of reaction-diffusion systems featuring the p-Laplacian operator, with a particular focus on systems incorporating Neumann boundary conditions and the potential inclusion of delay. These systems serve as powerful mathematical models that find wide-ranging applications across numerous scientific and practical domains. By investigating their global stability, our study addresses several critical aspects. First, the dynamics of such systems are intrinsic to a multitude of real-world phenomena, including the spread of infectious diseases, the diffusion of chemicals in biological tissues, and heat conduction in materials. The inclusion of the p-Laplacian operator, known for capturing nonlinear effects, allows for a more faithful representation of the complex interactions and diffusion processes inherent in these phenomena.

Second, global stability analysis plays a crucial role in epidemiological applications, particularly in the control and predictability of diseases. It ensures that equilibrium states within the systems are robust and that the dynamics tend to stabilize under various conditions. This insight has profound implications for optimizing processes, managing epidemics, controlling chemical reactions, and enhancing the reliability of materials and structural components, which is pertinent in different fields like engineering and others.

Furthermore, our approach to constructing Lyapunov functionals for PDEs based on those developed for ODEs represents a unifying and systematic method for analyzing and predicting the behavior of diverse dynamical systems. This methodological contribution not only advances the theoretical foundations of nonlinear systems