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The Dynamics of the Spruce Budworm-Bird System on Time Scale*

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Abstract We perform a geometric analysis focusing on relaxation oscillations and canard cycles within a singularly perturbed predator-prey system involving budworm and birds. The system undergoes a comprehensive stability analysis, leading to the identification of canard cycles in proximity to the Hopf bifurcation points. The study particularly highlights the transition from smaller Hopf-type cycles to larger relaxation cycles. And the expression of transition threshold $\mu_c(\sqrt{\varepsilon})$ of the spruce budworm-bird system is obtained innovatively. Furthermore, numerical simulations are carried out to validate the theoretical findings.

Keywords Slow-fast timescale, relaxation oscillation, canard cycle, predator-prey system

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1. Introduction

Relaxation oscillations represent a form of periodic solutions prevalent in slow-fast systems, extensively utilized in modeling chemical and biological processes. These oscillations manifest as sequential cycles characterized by alternating phases of dissipation and abrupt changes. Specifically, within the domain of relaxation oscillations lie the noteworthy phenomena of canard cycles, depicting trajectories within slow-fast systems that persist in proximity to a repelling slow manifold for a duration of $O(1)$ time [1]. Notably, recent years have witnessed a revitalized interest in the study of canard cycles, a resurgence intertwined with both the theoretical underpinnings of dynamical systems and their practical applications.

In the realm of smooth slow-fast systems, the emergence of a Hopf bifurcation occurs when the slow nullcline intersects the fast nullcline transversely, commonly near a fold or local extremum referred to as the critical manifold. This pivotal occurrence transpires particularly when the fast nullcline assumes an 'S'-shaped structure. Subsequently, the evolution of Hopf cycles gradually transforms into relaxation oscillations. This intriguing transition from Hopf cycles to relaxation oscillations takes place within an exponentially minute parameter range, recognized as a canard explosion [2,3].

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Around the era of 1980, the conception of geometric singular perturbation theory, pioneered by Fenichel [4], explicated the formation and dynamic analysis of canard explosions. This theory adopts a geometric approach in addressing singular perturbation quandaries, prominently involving the crucially significant normally hyperbolic condition. Noteworthy advancements have surfaced in recent years, introducing a prevalent mechanism aimed at scrutinizing instances where the condition of normal hyperbolicity fails. This innovative approach amalgamates blow-up techniques and dynamical systems, originally introduced by Dumortier and Roussarie [5], later harnessed by Krupa and Szmolyan [1].

In recent years, Brons et al. [6,7] conducted an analysis on mixed-mode oscillations arising from canard-type and singular Hopf bifurcations in a model representing a forest pest ecosystem. Izhikevich [8] explored bifurcations in both resting and active states of the burster, introducing a comprehensive theoretical classification method termed 'top-down' fast/slow dynamics bifurcation analysis. This methodology facilitated a meticulous examination and elucidation of diverse bifurcation mechanisms governing discharge rhythm in fast/slow neuron models [9,10].

Since the initial discovery of the correlation between canards and Mixed-Mode Oscillations (MMOs) in neuron models by [11], subsequent research consistently demonstrated and established the potential link between these phenomena [12,13]. A detailed theoretical exposition is provided in the literature [14,15].

It is well-acknowledged that predation can induce oscillations in interacting species, and numerous field and experimental data showcase periodic fluctuations in both predator and prey populations [16]. The existence of limit cycles in predator-prey models has been extensively investigated (see, for instance, [17]). Regarding the presence of relaxation dynamics, pertinent references include [18].

In the realm of biology and ecology, there is notable interest in exploring the dynamics of slow-fast prey-predator systems when the predator's death rate remains exceptionally low. For instance, predator animals like birds may only feed once every two or three hours due to low hunting success. Conversely, prey such as the spruce budworm reproduces rapidly, with a single female capable of producing tens of thousands of offspring in a short period. Their reproductive and mortality rates exhibit considerable variation over the same time frame. Hence, it is judicious to investigate slow-fast predator-prey systems incorporating small parameters.

In this paper, we study relaxation oscillations and canard cycles of the spruce budworm-bird system by combining geometric singular perturbation theory [4] and the Hopf bifurcation Theorem [1]. Specifically, in Section 2, the stability analysis of the system is performed. Then in Section 3 we introduce the slow-fast system. In Section 4 we discussed GSPT and blow-up technique for a detailed mathematical analysis of slow-fast systems. Finally, the conclusion of our work is drew in Section 5.

2. Model and its linear stability analysis

2.1. Predator-prey model

First, the dynamics of the budworm population without the action of predation will be described by the logistic equation

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = r\omega\left(1 - \frac{\omega}{K}\right),$$