Journal of Nonlinear Modeling and Analysis
Volume 7, Number 3, June 2025, 823–839
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.823

Three Weak Solutions for $(p(x),q(x))$-Biharmonic Problem with Hardy Weight with Two Parameters

Ibrahim Chamlal[1,†], Mohamed Talbi[2], Najib Tsouli[1] and Mohammed Filali[1]

Abstract The focus of this study is on the existence of three solutions to $(p(\cdot), q(\cdot))$-biharmonic operator with an $s(x)$-Hardy term under no-flux boundary conditions. Our method is based on the variational method and critical point results.

Keywords Variational method, singular problem, $p(x)$-biharmonic operator

MSC(2010) 35D30, 35G30, 35J75, 46E35.

1. Introduction

Singular elliptic problems have been intensively studied in the last decades, and have drawn attention in many types of contexts and applications, including heat conduction theory, boundary layer phenomena, biological pattern formation, morphogenesis and chemical heterogeneous catalysts (see [8, 32, 35, 36, 38, 41]).

In 2023 A. Khaleghi and A. Razani [26] studied the following $(p(x), q(x))$-biharmonic problem containing a singular term with exponent constant

$$\begin{cases} \Delta^2_{p(x)}u + \Delta^2_{q(x)}u + \theta(x)\frac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain with boundary of class C^1; $p, q \in C_+(\bar{\Omega})$, $\theta \in L^\infty(\Omega)$ is a real positive function, $1 < s < N/2$, λ is a positive parameter, and f is a Carathéodory function. They proved the existence and multiplicity of weak solutions of this problem, through the use of variational approaches and critical point results.

Also in [3], A. Ayoujil et al, examined a class of $(p_1(\cdot), p_2(\cdot))$-biharmonic of the form

$$\begin{cases} \Delta\left(|\Delta u|^{p_1(x)-2}\Delta u\right) + \Delta\left(|\Delta u|^{p_2(x)-2}\Delta u\right) = f(x,u) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega. \end{cases}$$

Several studies have focused on different equations in the $p-q-$laplacien oprator (see [9, 24, 28, 33, 43]). However in [40], Honghui Yin and Zuodong Yang, studied

†the corresponding author.
Email address: ibra4.chamlal@gmail.com (I. Chamlal), talbijdi@gmail.com (M. Talbi), n.tsouli@ump.ac.ma (N. Tsouli), filali1959@yahoo.fr (M. Filali)
[1]Department of Mathematics, Mohammed First University, Oujda, 60000, Morocco
[2]Department of Mathematics, Regional Centre of Trades Education and Training, Oujda, 60000, Morocco

the following problem.

$$\begin{cases} -\Delta_p u - \Delta_q u = \theta V(x)|u|^{r-2}u + |u|^{p^*-2}u + \lambda f(x,u), & x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{cases}$$

They showed that the problem has an infinite weak solution. Additionally, they obtained some results for the case $1 < q < p < r < p^*$, by using variational methods.

Weihua Wang [39] obtained multiple solutions for $\triangle_p^2 u = \frac{\mu|u|^{r-2}u}{|x|^s} + f(x,u)$ with Dirichlet boundary conditions, and the same problem with Navier boundary conditions, where $2 < 2p < N, p \le r < p^*(s) = \frac{(N-s)p}{N-2p} \le p^*(0) := p^*, \mu \ge 0$.

Over the past few years, there has been a lot of interest in the $p(x)$-biharmonic problem involving the $s(x)-$Hardy weight. It is the reason why this paper is a significant step in that direction. In this article, we focus on a particular class of singular fourth-order elliptic problems with no-flux boundary conditions.

$$(P_\lambda)\begin{cases} \Delta^2_{p(x)}u + \Delta^2_{q(x)}u + a(x)|u|^{p(x)-2}u = \mu m(x)\dfrac{|u|^{s(x)-2}u}{|x|^{s(x)}} + \lambda f(x,u) \text{ in } \Omega, \\ u = constant,\ \Delta u = 0, \qquad\qquad \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial}{\partial n}\Big(|\Delta u|^{p(x)-2}\Delta u\Big) + \frac{\partial}{\partial n}\Big(|\Delta u|^{q(x)-2}\Delta u\Big) ds = 0, \end{cases}$$

where $\Delta^2_{p(x)}u$ is the $p(x)$-biharmonic operator, $\Omega \subset \mathbb{R}^N (N > 2)$ is a smooth bounded domain and $0 \in \Omega$, λ is a real positive parameter, and the fonctions $p(x), q(x), r(x) \in C(\overline{\Omega})$.

We start by giving the assumptions that we will consider for our problem (PV).

H(r,q,p) $\quad 1 < q^- < q^+ < p^- < p^+ < \frac{N}{2}$,
where $\quad h^- := \min_{x\in\bar\Omega} h(x), \quad h^+ := \max_{x\in\bar\Omega} h(x)$.

(s) $p^+ < s^- \le s(x) < p_2^\star(x)$ for all $x \in \overline{\Omega}$, and $s^+ - \frac{1}{2} < s^-$.

(a) $a \in L^\infty(\Omega)$ and there exists $a_0 > 0$ such that $a(x) \ge a_0$ for all $x \in \Omega$.

(f_1)
$$|f(x,t)| \le a_1 + a_2|t|^{z(x)-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$
where $a_1,\ a_2 > 0$ and $1 < z(x) < p^-$, $\forall x \in \overline{\Omega}$.

(m) $m \in L^{\gamma(x)}(\Omega)$ is a changing sign function, where $\gamma \in C_+(\overline{\Omega})$ and $\frac{1}{p_2^\star(x)} + \frac{1}{\gamma(x)} < \frac{1}{s(x)}$ for all $x \in \overline{\Omega}$.

This paper has the following structure. In Section 2, we list a few standard definitions, fundamental properties, and background information on generalized Lebesgue-Sobolev spaces. In Section 3 under the case of the variable exponent, we prove the Sobolev-Hardy type compact embedding theorem and provide some preliminary results.

We get the existence of one weak solution nontrivial for the problem (P_λ) in Section 4. After that in Section 5, we show that the problem (P_λ) has two and three solutions.