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Existence and Multiplicity of Solutions for Anisotropic Discrete Boundary Value Problems Involving the $(p_1(t), p_2(t))$-Laplacian

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Abstract For an anisotropic discrete nonlinear problem with a variable exponent, we demonstrate both the existence and multiplicity of nontrivial solutions in this study. Our technique is based on variational methods.

Keywords Discrete nonlinear boundary value problems, nontrivial solution, $p(t)$-Laplacian, critical point theory

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1. Introduction

In this paper, we study the existence and multiplicity of nontrivial solutions for the following discrete anisotropic problem

$$(P)\quad \begin{cases} -\sum_{i=1}^{2} \Delta(|\Delta y(t-1)|^{p_i(t-1)-2}\Delta y(t-1)) = h(t, y(t)),\ t\in[1,N]_{\mathbb{Z}}, \\ y(0) = y(N+1) = 0, \end{cases}$$

where $N \geq 2$ is an integer, $[1, N]_{\mathbb{Z}}$ is the discrete interval given by $\{1, 2, 3..., N\}$, $\Delta y(t) = y(t+1) - y(t)$ is the forward difference operator, $h : [1, N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function in the second variable and $p_1, p_2 : [0, N]_{\mathbb{Z}} \longrightarrow [2, \infty[$.

As usual, a solution of (P) is a function $y : [0, N + 1]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ which satisfies both equations of (P).

We would like to point out that issue (P) is a discrete equivalent of the variable exponent anisotropic problem

$$\begin{cases} -\sum_{j=1}^{N}\sum_{i=1}^{2} \frac{\partial}{\partial x_j}\left(|\frac{\partial y}{\partial x_j}|^{p_{i,j}(x)-2}\frac{\partial y}{\partial x_j}\right) = h(x, y),\quad x\in\Omega, \\ y(x) = 0,\quad x\in\partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a smooth boundary, $h \in C\left(\overline{\Omega}\times\mathbb{R},\mathbb{R}\right)$ is a given function that satisfies certain properties, and $p_{i,j}(x)$ are continuous functions on $\overline{\Omega}$, with $p_{i,j}(x) \geq 2$ for $(i, j, x) \in [1, 2]_{\mathbb{Z}} \times [1, N]_{\mathbb{Z}} \times \Omega$.

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The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural networks, ecology, cybernetics, mechanical engineering, statistics, optimal control, electrical circuit analysis, population dynamics, biology and other fields; (see, for example [1, 21, 22, 26]). The existence and multiplicity of solutions to boundary value issues for difference equations with the $p(.)$-Laplacian operator have recently attracted more attention. Fixed point theorems in cones are typically used to get these results on this issue (see [3, 19, 23, 24] and references therein). It is widely recognized that, variational methods, critical point theory and also monotonicity methods are powerful tools to investigate the existence and multiplicity of solutions of various problems, (see the monographs [4, 6–12, 14–18, 20, 27–30]).

In this paper, we shall study the existence and multiplicity of nontrivial solutions of (P), via min-max methods and Mountain Pass Theorem.

To state our main results, we use the following notation:

$$p_i^+ = \max_{t\in[0,N]_{\mathbb{Z}}} p_i(t), \;\; p_i^- = \min_{t\in[0,N]_{\mathbb{Z}}} p_i(t), \quad \text{for } i = 1, 2;$$

$$p_{\max}^+ = \max\{p_1^+, p_2^+\}, \;\; p_{\max}^- = \max\{p_1^-, p_2^-\}, \;\; p_{\min}^- = \min\{p_1^-, p_2^-\}.$$

The following theorems are the key findings of this paper:

Theorem 1.1. *Assume that*

(H_1) there exists $\delta > 2^{p_{\max}^+ + 1}(N+1)^{\frac{p_{\max}^+}{2}}$ such that

$$\lim \inf_{|x|\to\infty} \frac{p_{\min}^- H(t,x)}{|x|^{p_{\max}^+}} \geq \delta, \; \forall t \in [1,N]_{\mathbb{Z}},$$

where

$$H(t,x) = \int_0^x h(t,s)ds \quad \text{for} \quad (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}.$$

Then the problem (P) has at least one solution.

Example 1.1. Let us consider a continuous function $h : [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by the formula

$$h(t,x) = 2^{p_{\max}^+ + 2t} \frac{p_{\max}^+}{p_{\min}^-} (N+1)^{\frac{p_{\max}^+}{2}} |x|^{p_{\max}^+ - 2} x.$$

Clearly

$$H(t,x) = \frac{2^{p_{\max}^+ + 2t}}{p_{\min}^-} (N+1)^{\frac{p_{\max}^+}{2}} |x|^{p_{\max}^+}.$$

It is easy to see that

$$\lim \inf_{|x|\to\infty} \frac{p_{\min}^- H(t,x)}{|x|^{p_{\max}^+}} = 2^{p_{\max}^+ + 2t} (N+1)^{\frac{p_{\max}^+}{2}} \geq 2^{p_{\max}^+ + 2} (N+1)^{\frac{p_{\max}^+}{2}} .$$

Then $H(t,x)$ satisfies the condition (H_1) with $\delta = 2^{p_{\max}^+ + 2} (N+1)^{\frac{p_{\max}^+}{2}}$.