Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 3, June 2025, 904–924 DOI:10.12150/jnma.2025.904

# Two Minimal Residual NHSS Iteration Methods for Complex Symmetric Linear Systems

Yikang Wang[1,†] and Pingping Zhang[1]

**Abstract** For the large sparse complex symmetric linear systems, by applying the minimal residual technique to accelerate a preconditioned variant of new Hermitian and skew-Hermitian splitting (P*NHSS) method and efficient parameterized P*NHSS (PPNHSS) method, we construct the minimal residual P*NHSS (MRP*NHSS) method and the minimal residual PPNHSS (MRPPNHSS) method. The convergence properties of the two iteration methods are studied. Theoretical analyses imply that the MRP*NHSS method and the MRPPNHSS method converge unconditionally to the unique solution. In addition, we also give the inexact versions of MRP*NHSS method and MRPPNHSS method and their convergence proofs. Finally, numerical experiments show the high efficiency and robustness of our methods.

**Keywords** Complex symmetric linear systems, minimal residual technique, inexact versions, convergence properties

**MSC(2010)** 65F10, 65F50

# 1. Introduction

In this paper, we consider iterative methods for solving the complex symmetric linear systems of the form:

$$Ax \equiv (W + iT)x = b, \tag{1.1}$$

where $i = \sqrt{-1}$ denotes the imaginary unit, and $W,\ T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite. Furthermore, we assume that $W$ is a symmetric positive definite matrix. In addition, $b \in \mathbb{C}^n$ is given and $x \in \mathbb{C}^n$ is what we need to get. The linear system (1.1) appears in many applications, such as fast Fourier transform-based solution of certain time-dependent PDEs [1], structural dynamics [2], diffuse optical tomography [3], molecular scattering [4], parabolic and hyperbolic problems [5], and so on.

In recent years, based on the Hermitian and skew-Hermitian splitting (HSS) of matrix $A$, i.e., $A = H(A) + S(A)$, with

$$H(A) = \frac{1}{2}(A + A^*) = W, \quad S(A) = \frac{1}{2}(A - A^*) = iT,$$

[†]the corresponding author.
Email address:wyk10071996@163.com (Yikang Wang)
[1]School of Science, Chongqing University of Posts and Telecommunications, Chongqing, 400065

Bai, Golub and Ng [6] introduced an HSS iteration method to approximate the solution of the system of linear equations. Nevertheless, a shifted skew-Hermitian linear system needs to be solved at each iteration step of the HSS iteration method. To improve this situation, Bai, Benzi and Chen [7] came up with a modified HSS (MHSS) method. If sparse triangular factorizations are used to solve the linear sub-systems involved at each step, the MHSS iteration method is likely to require considerably less storage than the HSS iteration method since only two triangular factors rather than three have to be computed and stored. To further increase efficiency of MHSS method, Bai et al. proposed the preconditioned MHSS (PMHSS) iteration method and applied it to the distributed control problems in [8, 9]. Since an arbitrary matrix $V$ is positive definite, $\alpha V + W$ and $\alpha V + T$ are both real symmetric positive definite, the two sub-systems involved in each step of the PMHSS iteration can be effectively solved either exactly by a sparse Cholesky factorization, or inexactly by PCG method. Moreover, the PMHSS iteration method converges to the unique solution of the system of linear equations (1.1) for any positive constant $\alpha$ and any initial guess $x^{(0)}$.

Subsequently, the authors of [10] raised a new HSS (NHSS) method to solve the non-Hermitian positive definite linear systems. The convergence analysis showed that the NHSS method converges to the unique solution if $\sigma_{\max} \leq \lambda_{\min}$, where $\sigma_{\max}$ is the maximum singular value of the matrix $S(A)$ and $\lambda_{\min}$ is the minimum eigenvalue of the matrix $H(A)$. Besides, numerical examples showed the NHSS method performs very well when the Hermitian part of the coefficient matrix is dominant. In 2018, based on the NHSS iteration method, Xiao, Wang and Yin [11] introduced a preconditioned variant of NHSS (P$^*$NHSS) and an efficient parameterized P$^*$NHSS (PPNHSS) iteration methods for solving a class of complex symmetric linear systems. They proved that these iterative sequences are convergent to the unique solution of the linear system for any initial guess under a loose restriction on the parameters $\alpha$ and $\omega$. Numerical results showed that the PPNHSS iteration method outperforms PMHSS, NHSS and a preconditioned variant of the generalized successive over-relaxation (PGSOR) [12] methods from the point of view of iterations and CPU times whether the experimental optimal parameters are used or not. More iteration methods for solving a class of complex symmetric linear systems, see [13–20].

Recently, Yang, Cao and Wu [21] proposed a minimum residual HSS (MRHSS) method to improve the efficiency of the HSS method by making use of the minimum residual technique to HSS iteration scheme. Numerical results revealed that the MRHSS method is much more effective than the HSS method. Then, Yang [22] improved the problem of inconvenient verification of the convergence of MRHSS. In order to avoid shifted skew-Hermitian linear system, Zhang, Yang and Wu [23, 24] further used the minimum residual technique to the MHSS iteration scheme and proposed the minimum residual MHSS (MRMHSS) method. Inspired by the above idea, we apply the minimum residual technique on P$^*$NHSS and PPNHSS methods and develop the minimum residual P$^*$NHSS (MRP$^*$NHSS) method and the minimum residual parameterized P$^*$NHSS (MRPPNHSS) method to improve the efficiency of the two methods of [11]. The corresponding convergence theories are also established.

Throughout this paper, we denote by $(x, y) = y^* x$ the Euclidean inner product for any complex vectors $x, y \in \mathbb{C}^n$, and denote by $\|x\| = \sqrt{(x, x)}$ the Euclidean norm for any complex vector $x \in \mathbb{C}^n$. For an arbitrary matrix $X \in \mathbb{C}^{n \times n}$, $\|X\|$