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# A Fixed Point Results for Multivalued Mappings in Hausdorff Fuzzy $b-$Metric Spaces and Applications

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**Abstract** In this paper, we are interested in proving a general fixed point theorem for multivalued mappings in fuzzy $b-$metric spaces. The results presented in this paper not only generalize the findings from [23], but also yield additional specific outcomes. We present an application to establish the existence of a solution to the integral equation, demonstrating the significance of our result.

**Keywords** Fuzyy metric space, fuzzy $b-$metric space, t-norm, fixed point, implicit relation

**MSC(2010)** Primary 47H10, Secondary 54H25.

## 1. Introduction

Zadeh [26] first introduced the concept of fuzzy sets in 1965. A fuzzy set $M$ in $X$ is a function with domain $X$ and values in $[0,1]$. Heilpern [16] introduced the notion of fuzzy maps and established some fixed-point theorems for them.

In 1975, Kramosil and Michalek [17] proposed the idea of a fuzzy distance between two elements of a nonempty set, using the concepts of a fuzzy set and a $t$-norm.

A binary operation $T:[0,1]\times[0,1]\to[0,1]$ is a continuous $t$-norm if it satisfies the following conditions: $T$ is continuous, associative and commutative, $T(a,1)=a$ for all $a\in[0,1]$ and for all $a,b,c,d\in[0,1]$ if $a\le c$ and $b\le d$ then $T(a,b)\le T(c,d)$.

Typical examples of a continuous $t$-norm are $T_p(a,b)=a.b$, $T_{min}(a,b)=\min\{a,b\}$ and $T_L(a,b)=\max\{a+b-1,0\}$. George and Veeramani [11] generalized the concept of fuzzy metric spaces introduced by Kramosil and Michalek [17]. Given a non-empty set $X$, and $T$ is a contnuous $t$-norm, the 3-tuple $(X,M,T)$ is said to be a fuzzy metric space [11], [12] if $M$ is a fuzzy set on $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ $t,u>0$ :

$$\begin{cases} 1)\ \ M(x,y,t)>0, \\ 2)\ \ M(x,y,t)=M(y,x,t)=1 \quad \text{iff} \quad x=y, \\ 3)\ \ M(x,z,t+u)\ge T(M(x,y,t),M(y,z,u)), \\ 4)\ \ M(x,y,.) \quad \text{is left continuous function from} \quad (0,\infty)\to[0,1]. \end{cases}$$

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In mathematics, the study of fixed point theory in metric spaces has several applications, especially in solving differential equations. Many authors have studied the new class of generalized metric space, known as b-metric space, introduced by Bakhtin [5] in 1989. For example, see [1]- [4], [6]- [9], [20]. The relationship between $b$-metric and fuzzy metric spaces is considered in [15]. Conversely, [24] introduced the concept of a fuzzy $b$-metric space, substituting the triangle inequality with a weaker one.

In this paper, we prove the existence and uniqueness of the fixed point in fuzzy b-metric spaces. We present an application to determine the existence and uniqueness of a solution to an integral equation, demonstrating the significance of our result.

Throughout this paper, $C(X)$ will denote the family of nonempty compact subsets of $X$. For all $A, B \in C(X)$ and for all $t > 0$, we define a function on $C(X) \times C(X) \times (0, \infty)$ by

$$H_M(A, B, t) = \min \left\{ \inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t) \right\},$$

$$\text{where } M(C, y, t) = \sup_{z \in C} M(z, y, t).$$

The fuzzy $b$-metric induces $H_M$, which we refer to as the Hausdorff fuzzy $b$-metric. The triplet $(C(X), H_M, T)$ is referred to as the Hausdorff fuzzy $b$-metric space.

We define also $\delta_M(A, B, t)$ as follows:

$$\delta_M(A, B, t) = \inf\{M(a, b, t), \quad a \in A \quad b \in B\}, \quad t > 0.$$

It follows immediately from the definition of $\delta_M$ that

$$\delta_M(A, B, t) = 1 \iff A = B = \{.\} \text{ and}$$

$$M(a, b, t) \geq \delta_M(A, B, t) \quad \forall a \in A \quad \forall b \in B, \quad t > 0.$$

## 2. Preliminary

**Definition 2.1** ( [24]). A 3-tuple $(X, M, T)$ is called a fuzzy $b$-metric space if $X$ is an arbitrary nonempty set, $T$ is a continuous $t$-norm, and $M$ is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the conditions for all $x, y, z \in X$, $t, u > 0$ and a given real number $s \geq 1$ :

- $(b_1)$ $M(x, y, t) > 0$,
- $(b_2)$ $M(x, y, t) = 1$ if and only if $x = y$,
- $(b_3)$ $M(x, y, t) = M(y, x, t)$,
- $(b_4)$ $M(x, z, s(t + u)) \geq T(M(x, y, t), M(y, z, u))$,
- $(b_5)$ $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

**Remark 2.1.** In this paper we will further use a fuzzy $b$-metric space in the sense of definition 2.1 with an additional condition $\lim_{t \to \infty} M(x, y, t) = 1$.

Note that every fuzzy metric space is a fuzzy $b$-metric space with $s = 1$. However, the following example illustrates that the converse need not hold true.