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Decay of Solutions to the Three-Dimensional Generalized Navier-Stokes Equation with Nonlinear Damping Term

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Abstract In this paper, we consider the three-dimensional generalized Navier-Stokes equation with a nonlinear damping term $|u|^{\beta-1}(\beta \geq 1)$. Firstly, utilizing the Fourier splitting method, we derive decay estimates for weak solutions to the equations when $\alpha = 0$ and $\beta = 1$, as well as when $0 < \alpha < \frac{3}{4}$ for any $\beta = 2$. Secondly, for $0 < \alpha < \frac{5}{4}$ and any $\beta > \max\{\frac{4\alpha}{3} + 1, 2\}$, we obtain the same result.

Keywords Generalized Navier-Stokes equations, damping term, Fourier splitting method

MSC(2010) 37L05, 35Q35, 35B40.

1. Introduction

In this paper, we investigate the decay of solutions to the following Cauchy problem for the incompressible generalized Navier-Stokes equations with a damping term $|u|^{\beta-1}u(\beta \geq 1)$:

$$\begin{cases} u_t + (u \cdot \nabla)u + \Lambda^{2\alpha}u + \nabla P + |u|^{\beta-1}u = 0, \\ \text{div } u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^3$ and $P = P(x,t) \in \mathbb{R}$ represent the unknown velocity field and the pressure, respectively. u_0 denotes the prescribed initial data satisfying $\text{div} u_0 = 0$. $\alpha \geq 0$, $\beta \geq 1$, are real parameters. $\Lambda^{2\alpha}$ is defined through Fourier transform (see [7])

$$\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi), \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx.$$

In recent studies concerning the well-posedness of Equation (1.1) with $\alpha = 1$, Cai et al. [1] employed the Galerkin approximation method to investigate properties of the system. Their findings revealed the existence of a weak solution for any $\beta > 1$. Furthermore, they determined that for $\beta \geq \frac{7}{2}$, the solution becomes a global strong solution, and it is unique for $\frac{7}{2} \leq \beta \leq 5$. Subsequently, Zhou [2] improved these results, establishing that the strong solution exists globally for $\beta \geq 3$, and it is

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unique for all $\beta \geq 1$. More recently, Cai et al. [8] have established the global existence and uniqueness of strong solutions for Equation (1.1) when $\alpha \geq \frac{5}{4}$ for $\beta \geq 1$ and when $\frac{1}{2} + \frac{2}{\beta} \leq \alpha \leq \frac{5}{4}$ for $\frac{8}{3} \leq \beta < +\infty$.

Recently, our attention has been drawn to the asymptotic behavior of the weak solutions of (1.1) with $\alpha = 1$. Through the refinement of the traditional Fourier splitting method, Jia et al. [3] provided the L^2 decay rate of the weak solutions, which holds for $\beta \geq \frac{10}{3}$. Additionally, Jiang and Zhu et al. [4,5] demonstrated that, if the initial condition satisfies $\|e^{\triangle t}u_0\|_{L^2} \leq C(1+t)^{-\mu}$ with $\mu > 0$, the weak solutions of (1.1) with $\beta \geq 3$ obey the bound $\|u(t)\|_{L^2} \leq C(1+t)^{\min\{-\mu, \frac{3}{4}\}}$. Yang et al. [6] further strengthened this result, showing that for $\beta \geq \frac{7}{3}$, the weak solutions satisfy $\|u(t)\|_{L^2} \leq C(1+t)^{-\alpha_0}$, where α_0 is defined as $\alpha_0 = \begin{cases} \min\{\mu, \frac{5}{4}\}, & \beta \in (\frac{7}{3}, 9] \\ \min\{\mu, \frac{3\beta-7}{4(\beta-5)}\}, & \beta \in (9, \infty) \end{cases}$.

Recently, Jiu et al. [10] derived decay estimates for weak solutions of the three-dimensional generalized Navier-Stokes equations. Motivated by [10], we aim to enhance the derived decay estimates of the solution of (1.1) through an iterative approach.

This paper focuses on the long-term behavior of the weak solutions to system (1.1) specifically in the scenario where $\alpha < \frac{5}{4}$. Our aim is to assess the influence of the damping term by utilizing techniques detailed in references [4,6,9,10]. To establish our primary findings, we shall employ the Fourier splitting technique introduced by Schonbek [9]. Our main results are given by the following theorems.

Theorem 1.1. *Let $\alpha = 0$ or $\beta = 1$. For $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, the system admits a weak solution such that*

$$\|u\|_2^2 \leq Ce^{-2t},$$

where the constant C only depends on $\|u_0\|_{L^2(\mathbb{R}^3)}$.

Theorem 1.2. *Let $0 < \alpha < \frac{3}{4}$, $\beta = 2$. For $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, the system admits a weak solution such that*

$$\|u\|_2^2 \leq C(1+t)^{-\frac{3}{2\alpha}},$$

where the constant C depends on α and $\|u_0\|_{L^2(\mathbb{R}^3)}$.

Theorem 1.3. *Let $0 < \alpha < \frac{5}{4}$, $\beta > \max\{\frac{4\alpha}{3} + 1, 2\}$. For $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, the system admits a weak solution such that*

$$\|u\|_2^2 \leq C(1+t)^{-\frac{3}{2\alpha}},$$

where the constant C depends on α, β and $\|u_0\|_{L^2(\mathbb{R}^3)}$.

This paper is organized as follows. In Section 2, we will give some notations and lemmas which will be used in the proof of our main Theorems. We will give the proof Theorem 1.1, 1.2 and 1.3 in Section 3.

2. Preliminaries

The following Gagliardo-Nirenberg inequality plays a very important role in our estimation.