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Normal Form for the 1:1 Resonance Problems for Delayed Reaction-Diffusion Systems*

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Abstract This article presents a direct method for calculating the normal form coefficients of a 1:1 resonant Hopf bifurcation in reaction-diffusion systems with time delay and Neumann boundary conditions. The formulas obtained in this paper can be easily implemented using a computer algebra system such as Maple or Mathematica.

Keywords Normal form, 1:1 resonance, Hopf bifurcation, delayed reaction-diffusion

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1. Introduction

Hopf and generalized Hopf bifurcations have been extensively studied by many researchers (e.g., see Ref. [1–5], and are associated with a pair of purely imaginary eigenvalues at an equilibrium. If the Jacobian of a system evaluated at a critical point involves two pairs of purely imaginary eigenvalues, the so-called "double-Hopf bifurcation may occur. Such bifurcations may exhibit more complicated and interesting dynamic behavior such as quasi-periodic motions on tori, and chaos (e.g., see Ref. [6–10]). A bifurcation is called non-resonant if the ratio of the two eigenvalues is not a rational number, otherwise it is called resonant. The most important resonance is the 1:1 non-semisimple case, in which the purely imaginary eigenvalues at criticality are assumed to be double and non-semisimple. This bifurcation has been presented as an open problem in Kopell and Howard [11] and in Guckenheimer and Holmes [7] and in Ref. [12–14]. To date there has been little research on the 1:1 resonant Hopf bifurcation.

The 1:1 resonance is important in a number of applications such as wind-induced oscillations of bundled conductors and aircraft longitudinal dynamics when the eigenvalues corresponding to a pair of elastic modes approach each other and the imaginary axis, see Ref. [15].

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Normal form theory is one of the basic methods for the study of non-linear dynamics such as the singularity, Hopf bifurcation and homoclinic and heteroclinic bifurcations. The theory of normal form is concerned with constructing a series of near identity non-linear transformations that simplify the non-linear systems as much as possible. With the aid of normal form theory, we may obtain a set of simpler differential equations, which is topologically equivalent to the original systems. Being "simpler" means that some non-linear terms may be eliminated from the original differential equations. Also the normal form for a 1:1 resonance Hopf bifurcation was expressed by some researchers see Ref. [12, 13, 16], but there are no explicit formulas relating the coefficients of the original system to those of the normal form.

The main attention of the paper is focused on developing a new and efficient computation of the normal forms for 1:1 resonant Hopf bifurcation. This bifurcation has linear codimension-3, and a centre subspace of dimension 4. With the help of the results presented in this paper, one can apply the analysis to any physical problem exhibiting a generalized Hopf bifurcation with non-semisimple 1:1 resonance.

The aim of this paper is two-fold: first, to present an explicit formula for the normal form of a generalized Hopf bifurcation with non-semisimple 1:1 resonance. Second, to use the results with those obtained to the vector field.

2. Decomposition of the phase space

In this section, we explore the decomposition behaviour of abstract reaction diffusion retarded functional differential equation with parameters in the phase space $\mathcal{C} = C([-\tau, 0]; X^m)$, described by

$$\dot{u}(t) = D\Delta u + L(\mu)u_t + F(u_t, \mu), \tag{2.1}$$

where $u_t \in \mathcal{C}$ is defined by $u_t(\theta) = u(t+\theta), -\tau \leq \theta \leq 0$, $\mu \in R^p$ is a parameter vector in a neighborhood V of zero. $L(\mu) : V \to L(\mathcal{C}, X^m)$ is C^k for $k \geq 3$ and $F : \mathcal{C} \times R^p \to X^m$ is $C^k (k \geq 2)$ with $F(0, \mu) = 0, DF(0, \mu) = 0$, $D = diag(d_1, d_2, \cdots d_m)$ and $d_i > 0$ for $i = 1, 2, \cdots m$. $X^m : \{(u_1, u_2, \cdots u_m^T) \in (H^2(0, l\pi))^m : \frac{\partial u_i}{\partial x}(0, t) = \frac{\partial u_i}{\partial x}(l\pi, t) = 0, i = 1, 2, \cdots m\}$ is the real-valued Hilbert space.

For Laplacian operator Δ, we have the following properties (see [17–20]).

(P1) $D\Delta$ generates a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on X^m with $|T(t)| \leq Me^{\omega t}(t \geq 0)$for some $M \geq 1, \omega \in R$ and $T(t)$ is a compact operator for $t > 0$;

(P2) The eigenfunctions $\{\beta_k^j : k \in N_0 = N \cup 0, j = 0, 1, \cdots m\}$ of $D\Delta$, with corresponding eigenvalues $\{\mu_k : k \in N_0\}$, form an orthonormal basis for X^m where $\mu_k = -\frac{k^2}{l^2}, k \in N_0$ and $\beta_k^j = \gamma_m e_j, \gamma_m = \frac{cos\frac{k}{l}}{x}||cos\frac{k}{l}x|| : k \in N_0, j = 0, 1, \cdots m$. Denote

$$< \nu(.), \beta_k > := \begin{pmatrix} < \nu(.), \beta_k^1 > \\ < \nu(.), \beta_k^2 > \\ \cdots \\ < \nu(.), \beta_k^m > \end{pmatrix}$$

for $\nu \in \mathcal{C}$ and $\beta_k = (\beta_k^1, \beta_k^2, \cdots \beta_k^m)$.